

# Off-shell Amplitudes in Superstring Theory

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## Abstract

Computing the renormalized masses and S-matrix elements in string theory, involving states whose masses are not protected from quantum corrections, requires defining off-shell amplitude with certain factorization properties. While in the bosonic string theory one can in principle construct such an amplitude from string field theory, there is no fully consistent field theory for type II and heterotic string theory. In this paper we give a practical construction of off-shell amplitudes satisfying the desired factorization property using the formalism of picture changing operators. We describe a systematic procedure for dealing with the spurious singularities of the integration measure that we encounter in superstring perturbation theory. This procedure is also useful for computing on-shell amplitudes, as we demonstrate by computing the effect of Fayet-Iliopoulos D-terms in four dimensional heterotic string theory compactifications using this formalism.

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# 1 Introduction

The usual formulation of critical string theories allows us to compute S-matrix elements of external states which do not suffer any mass renormalization. However generic states of string theory do undergo mass renormalization and for these states the usual string amplitudes do not compute S-matrix elements beyond tree level. The main reason for this is that the conformal invariance of the vertex operators requires us to set the momenta  $k_i$  carried by the external states to satisfy  $k_i^2 = -m_i^2$  where  $m_i$  is the *tree level* mass of the state. On the other hand computing S-matrix elements via the LSZ prescription requires us to impose the constraints  $k_i^2 = -m_{i,p}^2$  where  $m_{i,p}$  is the renormalized mass of the state. Thus if  $m_{i,p} \neq m_i$  there is an apparent conflict between the two conditions.

If we had an underlying string field theory then one could use this to define off-shell amplitudes and then use the standard LSZ prescription to compute S-matrix elements. Even in the absence of a string field theory one can give an ad hoc definition of off-shell amplitudes in string theory [1] (see also [2–8]). This has been fully developed in the context of bosonic string theory. The main problem with these amplitudes however is that the result depends on additional spurious data encoded in the choice of local coordinate system at the punctures

where the vertex operators are inserted. Since there is no canonical way of choosing these local coordinates the result for the off-shell amplitude is ambiguous.

The suggestion made in [9, 10] was to go ahead and compute the renormalized masses and S-matrix elements using these off-shell amplitudes despite the latter's dependence on spurious data, and then prove that the renormalized masses and the S-matrix elements computed this way do not depend on the spurious data. Refs. [9, 10] were able to establish the latter result provided we restrict the choice of local coordinates to within a special class – those satisfying the requirement of gluing compatibility. This means that if we are near a boundary of the moduli space where the punctured Riemann surface  $\Sigma$  used for computing an amplitude can be represented by two separate punctured Riemann surfaces  $\Sigma_1$  and  $\Sigma_2$  glued at one each of their punctures using the standard plumbing fixture prescription, then the choice of local coordinates at the external punctures of  $\Sigma$  must agree with those induced from the choice of local coordinates at the punctures of  $\Sigma_1$  and  $\Sigma_2$ . As long as we restrict the choice of local coordinates at the punctures within this class, the results for the physical quantities were shown to be independent of the choice of local coordinates.

For bosonic string theory we also have an underlying string field theory [11, 12]. Off-shell amplitudes computed from this field theory in the Siegel gauge fall within the general class of off-shell amplitudes described in [1], and automatically come with a set of gluing compatible coordinate system [13]. Thus the renormalized masses and S-matrix elements computed from string field theory would also agree with the ones computed with a general system of gluing compatible local coordinate system.

The discussion above should have an immediate generalization to supersymmetric string theories. The computation of on-shell amplitudes in this theory has undergone much clarification in recent years [14–19] where the notion of local coordinate systems on Riemann surfaces is replaced by local superconformal coordinates on super-Riemann surfaces and the final result for the amplitude is expressed as an integral over supermoduli spaces of super-Riemann surfaces instead of ordinary moduli spaces of Riemann surfaces. Generalizing the results of [9, 10] we would expect that the definition of an off-shell amplitude will now depend on the choice of local super-conformal coordinate system at the punctures, and that as long as the choice of the local superconformal coordinate system at the punctures is gluing compatible, the result for physical quantities should be independent of the choice of these coordinate systems. However the complete set of rules for off-shell amplitudes in superstring theory have not been laid out, although [15, 16] go a long way. Another route to defining off-shell amplitudes in superstring

theory would be to develop a superstring field theory. Despite considerable progress [20–31] this has not yet been fully achieved. This prevents us from carrying out explicit computation of renormalized masses and S-matrix elements in string theory except at low orders (*e.g.* at one loop order two point amplitude with *on-shell* external states is enough to compute the renormalized mass).

The goal of this paper is to give a definition of off-shell amplitude in superstring theory which can be used for practical computation. However instead of using superconformal formalism where the spurious data resides in the choice of local superconformal coordinates at the punctures, we shall use the formalism involving picture changing operators [32, 33] where the spurious data resides in the choice of local *bosonic* coordinate system at the punctures and the locations of the picture changing operators. Thus in this formalism the off-shell amplitudes are expressed as integrals over the moduli spaces of ordinary punctured Riemann surfaces, with the integrand being appropriate correlations functions of off-shell vertex operators, ghost fields and picture changing operators.

It has been known since [33] that the choice of locations of the picture changing operators corresponds to a choice of gauge for the gravitino field. It has also been known from the work of [19] that it is not possible to make a global choice of gauge for the gravitino field – we must work with different gauge choice in different parts of the moduli space. This breakdown of global gauge choice for the gravitino shows up in the picture changing formalism as spurious singularities of the integration measure appearing in a real codimension two subspace of the moduli space. We give a procedure for dealing with these singularities by introducing the notion of ‘vertical integration’ – integrating along a direction in which the location of the picture changing operators vary keeping the moduli fixed. The off-shell amplitude defined this way is ambiguous, but this ambiguity is at the same level as the one associated with the choice of locations of the picture changing operators and does not affect the renormalized masses or S-matrix elements.

The rest of the paper is organised as follows. In §2 we review the construction of off-shell amplitudes in bosonic string theory, following closely the work of [1, 12]. In §3 we generalize this construction to off-shell NS sector amplitudes in superstring theory using picture changing operators. We allow the locations of the picture changing operators to vary as we change the moduli of the Riemann surface. In this case we need to take into account the fact that the correlation function for computing the integration measure requires insertion of additional operators related to the picture changing operators by a set of descent equations [21, 33].

In §4 we discuss the origin of the spurious poles in the superstring integration measure and our method of dealing with them using the notion of vertical integration. In §5 we use this formalism to show that the renormalized masses and S-matrix elements of special states are independent of the choice of the locations of the picture changing operators even though the off-shell amplitudes do depend on them. In §6 we extend our prescription to amplitudes involving Ramond sector external states. This turns out to be more subtle than those involving NS sector external states and we suggest a way to deal with these subtleties by giving up manifest symmetry of the off-shell amplitude under the permutations of the external states. This will lead to a sensible approach if the S-matrix elements can be shown to have the permutation symmetry, but this has not been proved. Finally in §7 we illustrate the utility of our method even for on-shell amplitudes by applying it to compute the effects of Fayet-Iliopoulos terms in SO(32) heterotic string theory compactified on Calabi-Yau 3-folds [34]. This computation has been done earlier in different formalism [35–41], and our analysis using picture changing operator yields results in agreement with the earlier results.

We end this introduction with a word on convention. As emphasized in [9, 10], an off-shell amplitude  $\Gamma_{a_1 \dots a_n}^{(n)}(k_1, \dots k_n)$  in string theory – where  $a_i$  denotes the quantum numbers and  $k_i$  denotes the momentum of the  $i$ -th external state – do not compute the analog of the off-shell Green’s function  $G_{a_1 \dots a_n}^{(n)}(k_1, \dots k_n)$  in a quantum field theory. Instead the two are related as

$$\Gamma_{a_1 \dots a_n}^{(n)}(k_1, \dots k_n) = G_{a_1 \dots a_n}^{(n)}(k_1, \dots k_n) \prod_{i=1}^n (k_i^2 + m_i^2) \quad (1.1)$$

where  $m_i$  is the *tree level* mass of the  $i$ -th external state. Throughout this paper this is what we shall analyze. Of course once we have computed  $\Gamma^{(n)}$ , it is easy to find  $G^{(n)}$  using (1.1).

## 2 Off-shell amplitudes in the bosonic string theory

In this section we shall review the construction of off-shell amplitudes in bosonic string theory [1, 12] following closely the conventions of [12]. Let  $\mathcal{M}_{g,n}$  denote the moduli space of genus  $g$  Riemann surface with  $n$  punctures,  $\mathcal{P}_{g,n}$  denote the moduli space of genus  $g$  Riemann surface with  $n$  punctures with some choice of local coordinates around each puncture and  $\widehat{\mathcal{P}}_{g,n}$  denote the quotient of  $\mathcal{P}_{g,n}$  by independent phase rotation of the local coordinate around each puncture. Both  $\mathcal{P}_{g,n}$  and  $\widehat{\mathcal{P}}_{g,n}$  are infinite dimensional spaces. We also denote by  $\mathcal{M}_g$  the moduli space of genus  $g$  Riemann surface without the punctures. Then we have the natural projection

$$\mathcal{P}_{g,n} \rightarrow \widehat{\mathcal{P}}_{g,n} \rightarrow \mathcal{M}_{g,n} \rightarrow \mathcal{M}_g, \quad (2.1)$$

which corresponds to forgetting about some part of the data at each step. In fact we can regard  $\mathcal{P}_{g,n}$  to be a fiber bundle over the base  $\widehat{\mathcal{P}}_{g,n}$  with the phases of the local coordinates acting as the fiber directions,  $\widehat{\mathcal{P}}_{g,n}$  as the fiber bundle over the base  $\mathcal{M}_{g,n}$  with the choice of the local coordinates at the punctures up to phases as the fiber directions, and  $\mathcal{M}_{g,n}$  as the fiber bundle over the base  $\mathcal{M}_g$  with the locations of the punctures as the fiber directions.

## 2.1 Schiffer variation

Let  $\Sigma$  denote an element of  $\mathcal{P}_{g,n}$ , i.e. a Riemann surface of genus  $g$  and  $n$  punctures and some specific choice of local coordinates around each puncture. For given  $\Sigma$ , let  $w_a$  be the choice of local coordinate around the  $a$ -th puncture with the puncture situated at  $w_a = 0$ . We shall assume that the coordinates  $\{w_a\}$  have been chosen (possibly by scaling them with small numbers) so that  $w_a$  is a valid coordinate system on  $\Sigma$  for  $|w_a| \leq 1$ . We denote by  $D_a$  the disk  $|w_a| < 1$ . It will also be convenient to choose some fixed coordinate system on  $\Sigma - \cup_a D_a$ . A concrete way to do this is as follows. We can cut  $\Sigma - \cup_a D_a$  along  $3g - 3 + 2n$  homotopically non-trivial circles to divide  $\Sigma - \cup_a D_a$  into  $2g - 2 + n$  disjoint parts, each with the topology of a sphere with three holes. We can then label the  $i$ -th part  $\sigma_i$  by a complex coordinate  $z_i$  in which  $\sigma_i$  takes the form of a complex plane with three holes cut out of it. We shall denote by  $z$  the collection of the coordinates  $\{z_i\}$ . At the boundary circle separating two such components  $\sigma_i$  and  $\sigma_j$ , the coordinates  $z_i$  and  $z_j$  are related by some functional relation

$$z_i = f_{ij}(z_j) \tag{2.2}$$

where  $f_{ij}(z_j)$  is an analytic function that maps the common circle between  $\sigma_i$  and  $\sigma_j$  from the  $z_j$  plane to the  $z_i$  plane in a one to one fashion but could have singularities elsewhere. Furthermore the coordinates  $w_a$  labelling the local coordinates on the punctures are also related to the coordinate  $z$  of one of the components  $\sigma_i$  – that shares the boundary circle  $|w_a| = 1$  – by a functional relation of the form

$$z = f_a(w_a), \tag{2.3}$$

where  $f_a$  maps the circle  $|w_a| = 1$  to the corresponding circle in the  $z$ -plane in a one to one fashion but could have singularities both inside  $D_a$  as well as on  $\Sigma - D_a$ . Note that by an abuse of notation, in (2.3) we have labelled the coordinate of  $\sigma_i$  as  $z$ . Since there is always a unique  $\sigma_i$  that shares a boundary with  $D_a$ , this will not cause any confusion. The information about the moduli of the Riemann surface as well as the local coordinate system around the punctures

is then contained in the transition functions  $f_{ij}$  and  $f_a$ , although they are not all independent. For example an infinitesimal coordinate transformation of the form  $z_i \rightarrow z_i + \epsilon v(z_i)$  where  $v(z_i)$  is a non-singular vector field on  $\sigma_i$  will change the  $f_{ij}(z_j)$ 's and possibly some  $f_a(w_a)$  if the  $i$ -th component shares a boundary with  $D_a$ , but these changes do not change the moduli of the Riemann surface or the local coordinates around the punctures.

We'll need to study the tangent space of  $\mathcal{P}_{g,n}$  associated with deformations of the punctured Riemann surface and/or the choice of local coordinates around the punctures. There are various ways of describing this tangent space *e.g.* by infinitesimal deformations of the various functions  $f_{ij}(z_j)$  and  $f_a(w_a)$ , but one convenient way of doing this is via the Schiffer variation. The idea of Schiffer variation is that locally we can generate the full set of deformations in  $\mathcal{P}_{g,n}$  by deforming the functions  $f_a(w_a)$  keeping the  $f_{ij}(z_j)$ 's fixed. We shall now describe how it can be used to define a tangent to  $\mathcal{P}_{g,n}$ . Let us consider a deformation in  $\mathcal{P}_{g,n}$  labelled by an infinitesimal parameter  $\epsilon$  and let  $f_a^\epsilon$  be the deformed form of  $f_a$ . We introduce the coordinate system  $w_a^\epsilon$  via the relations

$$z = f_a^\epsilon(w_a^\epsilon). \quad (2.4)$$

We can combine (2.3) and (2.4) to get a relation between  $w_a^\epsilon$  and  $w_a$  of the form

$$w_a^\epsilon = (f_a^\epsilon)^{-1}(f_a(w_a)) = w_a + \epsilon v^{(a)}(w_a), \quad (2.5)$$

for some vector field  $v^{(a)}(w_a)$  that is non-singular around the  $|w_a| = 1$  curve but can have singularities away from it. Thus we can use the vector field  $v^{(a)}(w_a)$  to describe a tangent vector of  $\mathcal{P}_{g,n}$ . We can also use (2.3)-(2.5) to write

$$f_a^\epsilon(w_a) = f_a(w_a) - \epsilon v^{(a)}(z), \quad v^{(a)}(z) \equiv f'_a(w_a) v^{(a)}(w_a). \quad (2.6)$$

By an abuse of notation we have used the same symbol  $v^{(a)}(z)$  with changed argument to represent the vector field  $v^{(a)}(w_a)$  written in the  $z$  coordinate system.

In order to be more general let us consider such vector fields around each puncture and consider a deformation of the type given in (2.5) around each puncture. Together they describe a deformation of  $\mathcal{P}_{g,n}$  and hence a tangent vector of  $\mathcal{P}_{g,n}$  labelled by  $\vec{v} = (v^{(1)}, \dots, v^{(n)})$ . It is easy to verify that if  $\delta_{\vec{v}}$  denotes the tangent vector of  $\mathcal{P}_{g,n}$  generated by  $\vec{v}$ , then we have

$$[\delta_{\vec{v}_1}, \delta_{\vec{v}_2}] = \delta_{[\vec{v}_2, \vec{v}_1]}, \quad [\vec{v}_2, \vec{v}_1]^{(a)} \equiv \left( v_2^{(a)}(z) \partial_z v_1^{(a)}(z) - v_1^{(a)}(z) \partial_z v_2^{(a)}(z) \right). \quad (2.7)$$

Now we have the following general results (see *e.g.* section 7 of [12] for proofs of these results):



1. Let  $v(z)$  be a globally defined vector field on  $\Sigma$  that is holomorphic everywhere except possibly at the punctures. If  $v^{(a)}(z)$  is the restriction of  $v(z)$  to  $\partial D_a$ , then the deformations generated by  $\vec{v}(z) \equiv (v^{(1)}(z), \dots, v^{(n)}(z))$  can be removed by coordinate redefinition on  $\Sigma - \cup_a D_a$ , i.e. a redefinition of the coordinates  $z_k$  on  $\sigma_k$ . Thus  $\vec{v}(z)$  describes a vanishing tangent vector on  $\mathcal{P}_{g,n}$ .<sup>1</sup> This also works in the reverse direction, i.e. if the  $n$ -tuple of vector fields  $(v^{(1)}, \dots, v^{(n)})$  fail to extend holomorphically into  $\Sigma$  as a globally defined vector field away from the punctures then  $\vec{v}(z)$  does describe a non-trivial deformation on  $\mathcal{P}_{g,n}$ .
2. If  $\vec{v}(z)$  does not extend holomorphically into  $\Sigma - \cup_a D_a$ , but extends holomorphically into the  $D_a$ 's and vanish at the punctures, then it describes the same point in  $\mathcal{M}_{g,n}$  but deforms the choice of local coordinate system around the punctures.
3. If  $\vec{v}(z)$  does not extend holomorphically into  $\Sigma - \cup_a D_a$ , but extends holomorphically into the  $D_a$ 's and does not vanish at the punctures, then it describes the same point in  $\mathcal{M}_g$  but moves one or more of the punctures.
4. If  $\vec{v}(z)$  does not extend holomorphically into  $\Sigma - \cup_a D_a$ , and has poles at one or more punctures, then it describes a deformation on  $\mathcal{M}_g$ , i.e. changes the moduli of the underlying Riemann surface. Furthermore the complete set of complex deformations of  $\mathcal{M}_g$  can be obtained by choosing a set of  $3g - 3$  such vector fields with poles of order  $1, \dots, 3g - 3$  at any of the punctures.

In the following we shall continue to denote by  $z$  some fixed coordinate system on  $\Sigma - \cup_a D_a$  represented by the collection of the  $z_i$ 's, and by  $w_a$  the local coordinates around the punctures.

## 2.2 Surface states

A surface state  $\langle \Sigma |$  associated with a Riemann surface  $\Sigma$  with  $n$  punctures is a state in the dual space of the  $n$ -fold tensor product of the Hilbert space  $\mathcal{H}$  of the underlying CFT. It describes the state that is created on the boundaries of  $D_a$  by performing the functional integral over the fields of the CFT on  $\Sigma - \cup_a D_a$ . More precisely, if we consider a state  $|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle$  in  $\mathcal{H}^{\otimes n}$ , then

$$\langle \Sigma | (|\Psi_1\rangle \otimes \dots \otimes |\Psi_n\rangle) \quad (2.8)$$

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<sup>1</sup>Such a vector field will generate a deformation in a bigger space that contains information about not only the local coordinates around the punctures but also the coordinate system  $z_k$  on the  $\sigma_k$ 's.

describes the  $n$ -point correlation function on  $\Sigma$  with the vertex operator for  $|\Psi_a\rangle$  inserted at the  $a$ -th puncture using the local coordinate system  $w_a$  around that puncture. Thus  $\langle\Sigma|$  depends not only on the moduli labelling  $\mathcal{M}_{g,n}$  but also on the choice of local coordinates around the punctures.  $\langle\Sigma|$  satisfies the identity

$$\langle\Sigma|\sum_{r=1}^n Q_B^{(a)} = 0, \quad (2.9)$$

where  $Q_B^{(a)}$  is the BRST operator acting on the Hilbert space of states at the  $a$ -th puncture. Furthermore they also satisfy the identity (see *e.g.* [12]):

$$\begin{aligned} \delta_{\vec{v}}\langle\Sigma| &= -\langle\Sigma|T(\vec{v}), \\ T(\vec{v}) &\equiv \left( \sum_a \oint dw_a v^{(a)}(w_a) T^{(a)}(w_a) + \sum_a \oint d\bar{w}_a \bar{v}^{(a)}(\bar{w}_a) \bar{T}^{(a)}(\bar{w}_a) \right) \\ &= \left( \sum_a \oint dz v^{(a)}(z) T^{(a)}(z) + \sum_a \oint d\bar{z} \bar{v}^{(a)}(\bar{z}) \bar{T}^{(a)}(\bar{z}) \right), \end{aligned} \quad (2.10)$$

where  $\delta_{\vec{v}}$  is the tangent vector of  $\mathcal{P}_{g,n}$  associated with the Schiffer variation induced by the vector fields  $\vec{v}$ ,  $T^{(a)}$ ,  $\bar{T}^{(a)}$  are the stress tensor components acting on the Hilbert space of the  $a$ -th puncture, and the integration contour over  $w_a$  ( $\bar{w}_a$ ) runs in the anti-clockwise (clockwise) direction around each puncture and includes the usual  $1/2\pi i$  normalization factors so that  $\oint dw/w = \oint d\bar{w}/\bar{w} = 1$ . In going from the second to the third line we have used the fact that  $v^{(a)}(z)T^{(a)}(z)$  transforms as a one form under a coordinate transformation. Using (2.10) and the Virasoro commutation relations it is easy to verify that

$$[T(\vec{v}_1), T(\vec{v}_2)] = T([\vec{v}_2, \vec{v}_1]). \quad (2.11)$$

Note that  $\vec{v}$  and  $i\vec{v}$  describe independent deformations. Equivalently we can take  $\vec{v}$  and  $\vec{\bar{v}}$  to be independent.

### 2.3 Integration measure on $\mathcal{P}_{g,n}$

We now describe the construction of a  $p$ -form on  $\mathcal{P}_{g,n}$  that can be integrated over a  $p$ -dimensional subspace – henceforth referred to as an integration cycle. Let  $|\Phi\rangle$  denote some element of  $\mathcal{H}^{\otimes n}$ . Now by definition a  $p$ -form should generate a number when contracted with

$p$  tangent vectors of  $\mathcal{P}_{g,n}$  and this number should be anti-symmetric under the exchange of any pair of tangent vectors. Since tangent vectors are labelled by the  $n$ -tuple of vector fields  $\vec{v}$ , what we are looking for is a multilinear function of  $p$  such  $n$ -tuple of vector fields. Let  $V_1, \dots, V_p$  be  $p$  tangent vectors of  $\mathcal{P}_{g,n}$  and let  $\vec{v}_1, \dots, \vec{v}_p$  be the corresponding  $n$ -tuple vector fields. First we introduce an operator values  $p$ -form  $B_p$ , whose contraction with the tangent vectors  $V_1, \dots, V_p$  is given by

$$B_p[V_1, \dots, V_p] = b(\vec{v}_1) \cdots b(\vec{v}_p) \quad (2.12)$$

where  $b(\vec{v})$  is defined in the same way as  $T(\vec{v})$ :

$$\begin{aligned} b(\vec{v}) &\equiv \left( \sum_a \oint dw_a v^{(a)}(w_a) b^{(a)}(w_a) + \sum_a \oint d\bar{w}_a \bar{v}^{(a)}(\bar{w}_a) \bar{b}^{(a)}(\bar{w}_a) \right) \\ &\equiv \left( \sum_a \oint dz v^{(a)}(z) b^{(a)}(z) + \sum_a \oint d\bar{z} \bar{v}^{(a)}(\bar{z}) \bar{b}^{(a)}(\bar{z}) \right), \end{aligned} \quad (2.13)$$

$b, \bar{b}$  being the anti-ghost fields.  $B_p$  is clearly anti-symmetric under the exchange of a pair of  $\vec{v}_i$ 's. Then we define the  $p$ -form  $\Omega_p^{(g,n)}$  as [12]:

$$\Omega_p^{(g,n)}(|\Phi\rangle) = (2\pi i)^{-(3g-3+n)} \langle \Sigma | B_p | \Phi \rangle. \quad (2.14)$$

Ghost number conservation on the genus  $g$  Riemann surface tells us that if  $|\Phi\rangle$  carries total ghost number  $n_\Phi$  then we must have

$$n_\Phi - p = 6 - 6g, \quad (2.15)$$

in order for  $\Omega_p^{(g,n)}(|\Phi\rangle)$  to be non-zero.

Now it follows from our previous discussion that if there is a globally defined holomorphic vector field on the whole of  $\Sigma - \cup_a D_a$ , then adding to each  $\vec{v}_i$  an arbitrary multiple  $c_i$  of  $v$  describes the same set of tangent vectors in  $\mathcal{P}_{g,n}$ . One can show that this addition does not change the value of  $\Omega_p^{(g,n)}(|\Phi\rangle)$  given in (2.14). The proof of this uses the fact that such a deformation adds to  $b(\vec{v}_i)$  a term  $c_i \oint v(z) b(z) dz$  where the integration contour winds around all the punctures. We can now deform the contour in the interior of  $\Sigma$  and contract it to a point showing that the corresponding contribution vanishes.

This shows that  $\Omega_p^{(g,n)}$  indeed describes a  $p$ -form in  $\mathcal{P}_{g,n}$  and not in a bigger space that also keeps track of possible addition of globally defined vector fields to the  $v_i^{(a)}$ 's.

## 2.4 Restriction to $\widehat{\mathcal{P}}_{g,n}$

So far we have worked with states in the general Hilbert space  $\mathcal{H}$  of matter-ghost CFT. From now on we shall work with a restricted Hilbert space  $\mathcal{H}_0$  defined via the condition

$$|\Psi\rangle \in \mathcal{H}_0 \quad \text{if} \quad (b_0 - \bar{b}_0)|\Psi\rangle = 0, \quad (L_0 - \bar{L}_0)|\Psi\rangle = 0, \quad (2.16)$$

and take  $|\Phi\rangle$  to be an element of  $\mathcal{H}_0^{\otimes n}$ . For later use we shall also introduce the subspace  $\mathcal{H}_1$  containing off-shell states of ghost number two in the Siegel gauge

$$|\Psi\rangle \in \mathcal{H}_1 \quad \text{if} \quad |\Psi\rangle \in \mathcal{H}_0, \quad (b_0 + \bar{b}_0)|\Psi\rangle = 0, \quad \text{ghost number}(|\Psi\rangle) = 2. \quad (2.17)$$

The physical states which will appear as external states in S-matrix computation will be of this type.

One can show that for states satisfying (2.16) the following properties hold:

1. The  $p$ -form given in (2.14), contracted with a tangent vector of  $\mathcal{P}_{g,n}$  whose projection onto  $\widehat{\mathcal{P}}_{g,n}$  vanishes, vanishes. This follows from the fact that such a tangent vector represents a deformation in which local coordinates at the punctures change by phases. Using (2.13) we see that contracting such a tangent vector with the  $p$ -form will insert into the correlation function a  $b(\vec{v})$  that is a linear combination of  $\oint w_a dw_a b^{(a)}(w_a) - \oint \bar{w}_a d\bar{w}_a \bar{b}^{(a)}(\bar{w}_a) = b_0^{(a)} - \bar{b}_0^{(a)}$ . This vanishes by (2.16).
2. The  $p$ -form given in (2.14) remains unchanged if we move in  $\mathcal{P}_{g,n}$  along a direction that leaves its projection into  $\widehat{\mathcal{P}}_{g,n}$  unchanged. Since such a deformation corresponds to changing the local coordinates at the punctures by phases, we see from (2.10) that they correspond to insertions of a linear combination of  $\oint w_a dw_a T^{(a)}(w_a) - \oint \bar{w}_a d\bar{w}_a \bar{T}^{(a)}(\bar{w}_a) = L_0^{(a)} - \bar{L}_0^{(a)}$ . This vanishes by (2.16).

This essentially tells us that the  $p$ -form defined in (2.14) can be regarded as a  $p$ -form on  $\widehat{\mathcal{P}}_{g,n}$ .

## 2.5 BRST identity

We shall now describe an important identity that is used for proving many properties of the off-shell amplitude. Let us denote by  $|\Phi\rangle$  a state in  $\mathcal{H}_0^{\otimes n}$ . Then we have the identity

$$\Omega_p^{(g,n)}(Q_B|\Phi\rangle) = (-1)^p d\Omega_{p-1}^{(g,n)}(|\Phi\rangle), \quad (2.18)$$

where  $Q_B = \sum_{a=1}^n Q_B^{(a)}$ ,  $Q_B^{(a)}$  being the BRST operator acting on the  $a$ -th copy of  $\mathcal{H}$ . Since this is an important identity that needs to be generalized for superstring theories, we shall review its proof [12]. For this let  $\widehat{V}_1, \dots, \widehat{V}_p$  be a set of  $p$  tangent vectors of  $\widehat{\mathcal{P}}_{g,n}$ , and  $\Omega_p^{(g,n)}(\widehat{V}_1, \dots, \widehat{V}_p)$  be the contraction of these  $p$  tangent vectors with  $\Omega_p^{(g,n)}$ . Furthermore let  $\vec{v}_1, \dots, \vec{v}_p$  be the  $n$ -tuple vector fields associated with the tangent vectors  $\widehat{V}_1, \dots, \widehat{V}_p$ . Then by definition:

$$\begin{aligned} d\Omega_{p-1}^{(g,n)}(\widehat{V}_1, \dots, \widehat{V}_p) &= \sum_{i=1}^p (-1)^{i+1} \widehat{V}_i \Omega_{p-1}^{(g,n)}(\widehat{V}_1, \dots, \widehat{V}_i \dots, \widehat{V}_p) \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{i+j} \Omega_{p-1}^{(g,n)}([\widehat{V}_i, \widehat{V}_j], \widehat{V}_1, \dots, \widehat{V}_i \dots, \widehat{V}_j \dots, \widehat{V}_p), \end{aligned} \quad (2.19)$$

where  $/$  indicates that the corresponding entry is deleted from the list and the  $\widehat{V}_i$  in the first term on the right hand side has to be regarded as a differential operator involving derivative with respect to the coordinates of  $\widehat{\mathcal{P}}_{g,n}$ . Now using (2.7), (2.12), (2.14) and (2.19) we can translate (2.18) to

$$\begin{aligned} \langle \Sigma | b(\vec{v}_1) \dots b(\vec{v}_p) Q_B | \Phi \rangle &= \sum_{i=1}^p (-1)^{p+i+1} \delta_{\vec{v}_i} \langle \Sigma | b(\vec{v}_1) \dots \not{b}(\vec{v}_i) \dots b(\vec{v}_p) | \Phi \rangle \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{p+i+j} \langle \Sigma | b([\vec{v}_i, \vec{v}_j]) b(\vec{v}_1) \dots \not{b}(\vec{v}_i) \dots \not{b}(\vec{v}_j) \dots b(\vec{v}_p) | \Phi \rangle \\ &= \sum_{i=1}^p (-1)^{p+i} \langle \Sigma | T(\vec{v}_i) b(\vec{v}_1) \dots \not{b}(\vec{v}_i) \dots b(\vec{v}_p) | \Phi \rangle \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{p+i+j} \langle \Sigma | b([\vec{v}_i, \vec{v}_j]) b(\vec{v}_1) \dots \not{b}(\vec{v}_i) \dots \not{b}(\vec{v}_j) \dots b(\vec{v}_p) | \Phi \rangle. \end{aligned} \quad (2.20)$$

To prove this equation we begin with the left hand side and move  $Q_B$  to the extreme left picking up commutators on the way. When  $Q_B$  acts on  $\langle \Sigma |$  the result vanishes by eq.(2.9). So we only need to worry about the (anti-)commutators. Using the fact that

$$\{Q_B, b_{\vec{v}}\} = T_{\vec{v}}, \quad (2.21)$$

we can express this into a sum of  $p$  terms where in the  $i$ -th term the  $b(\vec{v}_i)$  is replaced by  $T(\vec{v}_i)$  and we pick an extra factor of  $(-1)^{p-i}$ . Next we move the  $T(\vec{v}_i)$  in the  $i$ -th term to the extreme left thereby generating the first set of terms on the right hand side of (2.20). However in that process we pick up commutators with  $b(\vec{v}_j)$ 's using the relation

$$[T(\vec{v}_i), b(\vec{v}_j)] = -b([\vec{v}_i, \vec{v}_j]), \quad (2.22)$$

and then move the  $b([\vec{v}_i, \vec{v}_j])$  factor to the extreme left. This generates an extra factor of  $(-1)^{j-1}$ . Finally exchanging the labels  $i$  and  $j$  we recover the second set of terms on the right hand side of (2.20).

## 2.6 General parametrization of tangent vectors

Even though the Schiffer variations are able to describe arbitrary tangent vectors in  $\widehat{\mathcal{P}}_{g,n}$ , and are the most convenient ones for deforming the local coordinate system around the punctures and the locations of the punctures on a fixed Riemann surface, they are not always the most convenient way of describing the variation of the moduli of the Riemann surface itself. A more general description of a tangent vector can be given by deforming the functions  $f_{ij}(z_j)$  introduced in §2.1. In this case by following the same procedure as in (2.6) we can introduce the relations

$$z_i = f_{ij}^\epsilon(z_j^\epsilon), \quad z_j^\epsilon = (f_{ij}^\epsilon)^{-1}(f_{ij}(z_j)) \equiv z_j + \epsilon v(z_j), \quad (2.23)$$

where  $v(z_j)$  is a vector field on the Riemann surface that is analytic inside an annulus containing the common boundary circle between  $\sigma_i$  and  $\sigma_j$ . Then the contraction of  $\Omega_p^{(g,n)}$  with the corresponding tangent vector is given by inserting into the correlation function a factor of

$$b(v) = \left( \oint dz_j v(z_j) b(z_j) + \oint d\bar{z}_j \bar{v}(\bar{z}_j) \bar{b}(\bar{z}_j) \right) \quad (2.24)$$

with the integration contour over  $z_j$  ( $\bar{z}_j$ ) running along the circle forming the common boundary of  $\sigma_i$  and  $\sigma_j$  keeping the  $\sigma_j$  component to its left (right). This is the generalization of the statement that the contour of integration over  $w_a$  ( $\bar{w}_a$ ) in (2.13) was anti-clockwise (clockwise) in the  $w_a$  plane.

The simplest example of this is the plumbing fixture relation:

$$zw = q, \quad (2.25)$$

where  $q$  is a complex parameter labelling the ‘plumbing fixture variable’. The  $z$  coordinate system is used in the region  $|z| \geq |q|^{1/2}$  and the  $w$  coordinate system is used in the region  $|w| \geq |q|^{1/2}$ . If we regards  $q$  and  $\bar{q}$  as independent variables and consider the tangent vector  $\partial/\partial q$  then the corresponding vector field  $v(z)$  computed from (2.6) is given by  $-q^{-1}z$ . Thus the contraction of  $\Omega_p^{(g,n)}$  with such a vector will insert

$$-q^{-1} \oint dz z b(z) \quad (2.26)$$

into the correlation function. The contour runs anti-clockwise around  $z = 0$  along  $|z| = |q|^{1/2}$ . By appropriate deformation of the integration contour this definition of  $\Omega_p^{(g,n)}$  can be shown to be equivalent to the one given in terms of Schiffer variation.

## 2.7 Off-shell amplitude and gluing compatibility

So far we have described the construction of natural  $p$ -forms on  $\widehat{\mathcal{P}}_{g,n}$  for a given set of external states in  $\mathcal{H}^{\otimes n}$ . If we restrict the external states at the punctures by requiring each of them to have ghost number 2, so that the corresponding state  $|\Psi_1\rangle \otimes \cdots \otimes |\Psi_n\rangle$  carries total ghost number  $2n$ , then (2.15) tells us that (2.14) vanishes unless

$$p = 6g - 6 + 2n. \quad (2.27)$$

This is exactly the correct dimension of the moduli space of genus  $g$  Riemann surfaces. However for off-shell external states the  $6g - 6 + 2n$ -form defined in (2.14) does not descend down to a  $6g - 6 + 2n$ -form on  $\mathcal{M}_{g,n}$  since it depends on the choice of local coordinate system at the punctures and has non-vanishing contraction with tangent vectors which correspond to deformation of the local coordinates without any deformation of  $\mathcal{M}_{g,n}$ . Thus the best we can do is to regard  $\widehat{\mathcal{P}}_{g,n}$  as a fiber bundle over  $\mathcal{M}_{g,n}$  and integrate this  $(6g - 6 + 2n)$ -form over a section of the fiber bundle. This defines the off-shell string amplitude for external states  $|\Psi_1\rangle, \dots, |\Psi_n\rangle$ . The result depends on the choice of the section, reflecting the fact that the off-shell amplitudes depend on the choice of local coordinate system around the puncture. However the physical quantities like the renormalized masses and S-matrix elements are independent of the choice of the section [9, 10].

As discussed in [9, 10], for consistent off-shell amplitudes we need to impose on the choice of this section the requirement of gluing compatibility. This says that near a boundary of the moduli space when a genus  $g$  surface with  $n$ -punctures degenerates into a genus  $g_1$  surface with  $n_1$  punctures and a genus  $g_2$  surface with  $n_2$  punctures with  $g = g_1 + g_2$  and  $n = n_1 + n_2 - 2$ , the choice of local coordinates on the original surface must be taken to be those induced from the local coordinates at the punctures on the two surfaces into which it degenerates. More precisely suppose that  $u_1, \dots, u_{n_1}$  denote the local coordinates around the  $n_1$  punctures of the genus  $g_1$  Riemann surface and  $v_1, \dots, v_{n_2}$  denote the local coordinates around the  $n_2$  punctures of the genus  $g_2$  Riemann surface. Suppose further that we construct the genus  $g_1 + g_2$  Riemann surface by gluing the  $a$ -th puncture of the first surface and the  $b$ -th puncture of the second

surface using the plumbing fixture relation:

$$u_a v_b = e^{-s+i\theta}, \quad 0 \leq s < \infty, \quad 0 \leq \theta < 2\pi. \quad (2.28)$$

This will automatically give a choice of local coordinates on the genus  $g_1 + g_2$  Riemann surface with  $n_1 + n_2 - 2$  punctures. The requirement is that *for all Riemann surfaces of genus  $g = g_1 + g_2$  with  $n = n_1 + n_2 - 2$  punctures which can be constructed this way, the local coordinates at the punctures must be taken to be the ones that is induced from the choices  $(u_1, \dots, u_{n_1}, v_1, \dots, v_{n_2})$* . The off-shell Siegel gauge amplitudes constructed from closed string field theory automatically induces such gluing compatible local coordinate system [13].

A more explicit description of this condition is as follows. Let us describe the first Riemann surface as a collection of different components  $\{\sigma_k^{(1)}\}$  and  $D_1^{(1)}, \dots, D_{n_1}^{(1)}$  as in §2.1 and the second Riemann surface as a collection of different components  $\{\sigma_k^{(2)}\}$  and  $D_1^{(2)}, \dots, D_{n_2}^{(2)}$ . The choice of the sections in  $\widehat{\mathcal{P}}_{g_1, n_1}$  and  $\widehat{\mathcal{P}}_{g_2, n_2}$  correspond to specific relations between the coordinate systems on these different components on their common boundary circles. Now one of the components  $\sigma_k^{(1)}$ , which has common boundary with  $D_a^{(1)}$ , is glued to one of the components  $\sigma_k^{(2)}$ , having common boundary with  $D_b^{(2)}$ , according to (2.28) to form the Riemann surface of genus  $g$  and  $n$  punctures. For such Riemann surfaces we can label the coordinates of  $\mathcal{M}_{g, n}$  by the coordinates of  $\mathcal{M}_{g_1, n_1}$ , coordinates of  $\mathcal{M}_{g_2, n_2}$ ,  $s$  and  $\theta$ . On the other hand the coordinate of  $\mathcal{P}_{g, n}$  can be described by specifying the relationship between the coordinate systems on  $\{\sigma_k^{(1)}\}$ ,  $\{\sigma_k^{(2)}\}$ ,  $D_1^{(1)}, \dots, D_{n_1}^{(1)}$  and  $D_1^{(2)}, \dots, D_{n_2}^{(2)}$  on their overlap circles. Then the gluing compatibility condition requires that the section in  $\widehat{\mathcal{P}}_{g, n}$  should be chosen such that the relationships between the coordinates of  $\{\sigma_k^{(1)}\}$  and  $D_1^{(1)}, \dots, D_{n_1}^{(1)}$  depend only on a subset of the base coordinates – those labelling  $\mathcal{M}_{g_1, n_1}$  – but not on the coordinates of  $\mathcal{M}_{g_2, n_2}$  or  $s, \theta$ . Furthermore the dependence of these relations on the coordinates of  $\mathcal{M}_{g_1, n_1}$  must be the one induced from the choice of the section in  $\widehat{\mathcal{P}}_{g_1, n_1}$ . Similarly the relationships between the coordinate systems on  $\sigma_k^{(2)}$  and  $D_1^{(2)}, \dots, D_{n_2}^{(2)}$  depend only on the coordinates of  $\mathcal{M}_{g_2, n_2}$  according to the choice of the section in  $\widehat{\mathcal{P}}_{g_2, n_2}$ .

Let us denote by  $S_1$  and  $S_2$  a pair of sections on  $\widehat{\mathcal{P}}_{g_1, n_1}$  and  $\widehat{\mathcal{P}}_{g_2, n_2}$  and let  $S$  be the  $(6g - 6 + 2n)$  dimensional subspace of  $\widehat{\mathcal{P}}_{g, n} = \mathcal{P}_{g_1 + g_2, n_1 + n_2 - 2}$  containing the family of Riemann surfaces obtained by plumbing fixture of the Riemann surfaces associated with the sections  $S_1$  and  $S_2$ . Then the tangent space of  $S$  can be labelled by  $\partial/\partial s$ ,  $\partial/\partial \theta$  and the tangent vectors of  $S_1$  and  $S_2$ . It follows that the  $b$ -ghost insertions needed for computing the contraction of  $\Omega_{6g - g + 2n}^{(g, n)}$



with these tangent vectors automatically factorize into

$$-i B_{6g_1-6+2n_1}^{(1)} b_0^+ b_0^- B_{6g_2-6+2n_2}^{(2)}, \quad (2.29)$$

where the superscripts (1) and (2) refer to the two Riemann surfaces, and

$$b_0^\pm \equiv (b_0 \pm \bar{b}_0), \quad b_0 \equiv \oint du_a u_a b(u_a), \quad \bar{b}_0 \equiv \oint d\bar{u}_a \bar{u}_a \bar{b}(\bar{u}_a). \quad (2.30)$$

In (2.29)  $B^{(1)}$  and  $B^{(2)}$  represent the effect of contraction of  $\Omega_{6g-6+2n}^{(g,n)}$  with the tangent vectors of  $S_1$  and  $S_2$ ,  $b_0^+$  represents the effect of contraction of  $\Omega_{6g-6+2n}^{(g,n)}$  with the tangent vector  $\partial/\partial s$  and  $-i b_0^-$  represents the effect of contraction of  $\Omega_{6g-6+2n}^{(g,n)}$  with the tangent vector  $\partial/\partial \theta$ . Furthermore, it follows from (2.29), and the factorization property of correlation functions in conformal field theories on Riemann surfaces, that the full integration measure  $\Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle)$ , restricted to  $S$ , also factorizes. We shall begin with a more general factorization formula and then restrict to the case of interest. If  $|\Phi\rangle \in \mathcal{H}_0^{\otimes n}$  has the form  $|\Phi_1\rangle \otimes |\Phi_2\rangle$  where  $|\Phi_1\rangle \in \mathcal{H}_0^{n_1-1}$  denotes the states at the external punctures of the first Riemann surface and  $|\Phi_2\rangle \in \mathcal{H}_0^{n_2-1}$  denotes the states at the external punctures of the second Riemann surface, and if  $N_1$  and  $N_2$  denote total ghost numbers carried by  $|\Phi_1\rangle$  and  $|\Phi_2\rangle$ , then we have,

$$\begin{aligned} \Omega_p^{(g,n)}(|\Phi\rangle)|_S &= \frac{1}{2\pi} \sum_{\substack{0 \leq p_1 \leq 6g_1-6+2n_1, \ 0 \leq p_2 \leq 6g_2-6+2n_2 \\ p_1+p_2=p-2}} \sum_{i,j} \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ &\quad (-1)^{p_1 p_2 + N_1 + p_1 + 1} ds \wedge d\theta \wedge \Omega_{p_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{p_2}^{(g_2, n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2} \end{aligned} \quad (2.31)$$

where the subscript  $S$  on the left hand side denotes that this relation is valid for  $\Omega_p^{(g,n)}$  restricted to the section  $S$  and the subscripts  $S_1$  and  $S_2$  on the right denotes similar restrictions on  $\Omega_{p_1}^{(g_1, n_1)}$  and  $\Omega_{p_2}^{(g_2, n_2)}$ . The sum over  $i, j$  runs over all states in  $\mathcal{H}_0$  and  $\langle \varphi_i^c |$  is the conjugate state of  $|\varphi_i\rangle$  satisfying

$$\langle \varphi_i^c | \varphi_j \rangle = \delta_{ij}, \quad \langle \varphi_j | \varphi_i^c \rangle = (-1)^{n_{\varphi_i}} \delta_{ij}, \quad \sum_i |\varphi_i\rangle \langle \varphi_i^c| = (-1)^{n_{\varphi_i}} \sum_i |\varphi_j^c\rangle \langle \varphi_j| = \mathbf{1}, \quad (2.32)$$

where  $\langle \varphi_i |$  is the BPZ conjugate of  $|\varphi_i\rangle$  and  $n_{\varphi_i}$  is its ghost number. It follows that  $|\varphi_i^c\rangle$  does not belong to  $\mathcal{H}_0$ , but has the form  $(c_0 - \bar{c}_0)|\chi_i\rangle$  for some state  $|\chi_i\rangle \in \mathcal{H}_0$ . In (2.31) one factor of  $-1$  comes from the combination of  $-i$  in (2.29) and the normalization factor  $(2\pi i)^{-3g+3-n}$  in (2.14). Other sign factors come from rearranging the  $b$ -ghost insertions and external operator

insertions in proper order so as to admit the interpretation given on the right hand side of (2.31). In particular a factor of  $(-1)^{p_2 N_1}$  arise from the need to move the  $p_2$  number of  $b$ -ghost insertions through  $|\Phi_1\rangle$  so that they sit next to the external vertex operators inserted on  $\Sigma_2$ . Another factor of  $(-1)^{N_2+p_2}$  comes from noting that on the right hand side we have once used the last identity of (2.32) and that the  $|\varphi_j\rangle$  involved in this sum has  $n_{\varphi_j} = 6 - 6g_2 - N_2 + p_2$ . Finally a factor of  $(-1)^{(p_2+N_2)p_2}$  comes from moving  $\varphi_j$  through the  $p_2$  ghost insertions to sit next to the states  $|\Phi_2\rangle$ . Together they give the net factor of

$$(-1)^{p_2 N_1 + (N_2 + p_2)(p_2 + 1) + 1}. \quad (2.33)$$

Using ghost charge conservation we see that in order to get a non-vanishing contribution to the right hand side of (2.31) we need  $(-1)^{N_2+p_2} = (-1)^{N_1+p_1}$ . This reduces (2.33) to  $(-1)^{p_1 p_2 + N_1 + p_1 + 1}$  as given in (2.31). Eventually integration over  $\theta$  imposes a projection  $\delta_{L_0, \bar{L}_0}$  on  $|\varphi_j^c\rangle$ , and cancels the multiplicative factor of  $1/2\pi$ .

Another useful formula expresses  $\Omega_p^{(g,n)}|_S$  restricted to the boundary  $s = \Lambda$  for some large number  $\Lambda$ . We have to follow the same logic as before except that there are no  $ds$  factor in the wedge product and no  $b_0^+$  insertion in the correlator. The result is

$$\begin{aligned} \Omega_p^{(g,n)}(|\Phi\rangle)|_{S;s=\Lambda} &= \frac{1}{2\pi} \sum_{\substack{0 \leq p_1 \leq 6g_1 - 6 + 2n_1, \\ p_1 + p_2 = p-1}} \sum_{\substack{0 \leq p_2 \leq 6g_2 - 6 + 2n_2 \\ i,j}} \langle \varphi_i^c | b_0^- e^{-\Lambda(L_0 + \bar{L}_0)} e^{i\theta(L_0 - \bar{L}_0)} | \varphi_j^c \rangle \\ &\quad (-1)^{p_1 p_2 + p_2} d\theta \wedge \Omega_{p_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{p_2}^{(g_2, n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2}. \end{aligned} \quad (2.34)$$

The overall sign is calculated as follows. We begin with the configuration where  $d\theta$  sits to the extreme left in the wedge product and  $b_0^-$  sits to the extreme left in the correlation function. Following the same manipulation as before we get the sign factor given in (2.33), but now there is an extra factor of  $(-1)^{N_1+p_1}$  in order to move the  $b_0^-$  from the extreme left through the  $b$ -insertions associated with  $\Omega_{p_1}^{(g_1, n_1)}$  and  $|\Phi_1\rangle$ . (This factor was absent in the previous case since what was moved is  $b_0^+ b_0^-$ .) Now using the fact that ghost charge conservations demands that  $(-1)^{N_1+p_1} = (-1)^{N_2+p_2+1}$  we get the sign factor given in (2.34).

When each of the states at the external punctures describe Siegel gauge off-shell states carrying ghost number two, i.e. belong to the subspace  $\mathcal{H}_1$ , then  $N_1$  and  $N_2$  are even. If we further restrict to the case  $p_i = 6g_i - 6 + 2n_i$ , we get from (2.31)

$$\Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle)|_S = -\frac{1}{2\pi} \sum_{i,j} ds \wedge d\theta \wedge \Omega_{6g_1-6+2n_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{6g_2-6+2n_2}^{(g_2, n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2}$$

$$\times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0 + \bar{L}_0)} e^{i\theta(L_0 - \bar{L}_0)} | \varphi_j^c \rangle \quad (2.35)$$

Since in this case  $|\Phi_1\rangle$  has total ghost number  $2(n_1 - 1)$  and  $|\Phi_2\rangle$  has total ghost number  $2(n_2 - 1)$ , it follows from the ghost number conservation law given in (2.15) that  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  must also carry ghost number two. Furthermore the appearance of  $b_0^+ b_0^- = 2\bar{b}_0 b_0$  in the second line of (2.35) implies that the basis states  $|\varphi_i^c\rangle, |\varphi_j^c\rangle$  can be taken to be annihilated by  $c_0, \bar{c}_0$  and hence the conjugate basis states  $|\varphi_i\rangle, |\varphi_j\rangle$  should be annihilated by  $b_0, \bar{b}_0$ . Thus  $|\varphi_i\rangle$  actually belongs to the space  $\mathcal{H}_1$  described in (2.17). By normalizing the basis states  $|\varphi_i\rangle$  to satisfy

$$\langle \varphi_i | c_0^- c_0^+ | \varphi_j \rangle = \delta_{ij}, \quad c_0^\pm \equiv \frac{1}{2}(c_0 \pm \bar{c}_0), \quad (2.36)$$

we can ensure that

$$\langle \varphi_i^c | b_0^+ b_0^- | \varphi_j^c \rangle = \delta_{ij}. \quad (2.37)$$

Thus we can drop the second line of (2.35), insert the operator  $e^{-s(L_0 + \bar{L}_0)} e^{i\theta(L_0 - \bar{L}_0)}$  in front of  $|\varphi_i\rangle$  in the first line [9] and set  $j = i$  in the sum.

### 3 Off-shell NS sector amplitudes in superstring theory

In this section we shall construct off-shell amplitudes for NS sector external states in heterotic string theory by generalizing the procedure described in [21]. Ramond sector states require special treatment and will be discussed in §6. Generalization to type II string theory is straightforward and will be discussed briefly in §3.10. Also in this section we shall assume that it is possible to choose a gauge for the gravitino globally over the whole moduli space. In this case we can integrate out the fermionic coordinates of the supermoduli space and express the amplitudes as integrals over the moduli space of ordinary Riemann surfaces at the cost of inserting a set of picture changing operators into the world-sheet correlation functions [33]. As emphasized in [14], this assumption fails at sufficiently high genus. In such cases the measure in the moduli space, constructed using the picture changing operators, has spurious singularities in codimension two subspaces of the moduli space where the gauge choice for the gravitino breaks down [33]. In §4 we shall address how to deal with these spurious singularities.

#### 3.1 Superconformal ghost system and off-shell string states

A classical background in heterotic string theory is based on a two dimensional superconformal field theory with supersymmetry in the right-moving (holomorphic) sector of the world-sheet.

Besides the matter superconformal field theory with central charge (26,15), it also has anti-commuting  $b, c, \bar{b}, \bar{c}$  ghosts and commuting  $\beta, \gamma$  ghosts with total central charge  $(-26, -15)$ . The  $(\beta, \gamma)$  system can be ‘bosonized’ as

$$\gamma = \eta e^\phi, \quad \beta = \partial \xi e^{-\phi}, \quad \delta(\gamma) = e^{-\phi}, \quad \delta(\beta) = e^\phi, \quad (3.1)$$

where  $\xi, \eta$  are fermions and  $\phi$  is a scalar with background charge. We shall use the standard convention in which the (ghost number, picture number, GSO) assignments of various fields are:

$$\begin{aligned} c, \bar{c} : (1, 0, +), \quad b, \bar{b} : (-1, 0, +), \quad \gamma : (1, 0, -), \quad \beta : (-1, 0, -), \\ \xi : (-1, 1, +), \quad \eta : (1, -1, +), \quad e^{q\phi} : (0, q, (-1)^q). \end{aligned} \quad (3.2)$$

$e^{\pm\phi}$  are fermionic operators. The operator products of  $b, c, \xi, \eta$  and  $e^{q\phi}$  operators take the form

$$c(z)b(w) = (z-w)^{-1} + \dots, \quad \xi(z)\eta(w) = (z-w)^{-1} + \dots, \quad e^{q_1\phi(z)}e^{q_2\phi(w)} = (z-w)^{-q_1q_2}e^{(q_1+q_2)\phi(w)} + \dots. \quad (3.3)$$

With this  $\gamma(z)\beta(w) \sim -(z-w)^{-1}$ . This has an additional  $-$  sign compared to the standard convention used *e.g.* in [14], but we have chosen to stick to the bosonization rules (3.1) of [33]. The BRST charge is given by

$$Q_B = \oint dz j_B(z) + \oint d\bar{z} \bar{j}_B(z), \quad (3.4)$$

where

$$\bar{j}_B(\bar{z}) = \bar{c}(\bar{z})\bar{T}_m(\bar{z}) + \bar{b}(\bar{z})\bar{c}(\bar{z})\bar{\partial}\bar{c}(\bar{z}), \quad (3.5)$$

$$j_B(z) = c(z)(T_m(z) + T_{\beta,\gamma}(z)) + \gamma(z)T_F(z) + b(z)c(z)\partial c(z) - \frac{1}{4}\gamma(z)^2b(z). \quad (3.6)$$

Here  $\bar{T}_m(\bar{z})$  is the anti-holomorphic part of the matter stress tensor,  $T_m(z)$  is the holomorphic part of the matter stress tensor,  $T_{\beta,\gamma}(z)$  is the stress tensor of the  $(\beta, \gamma)$  system and  $T_F(z)$  is the world-sheet supersymmetry current in the matter sector. The signs of various terms in (3.6) are consistent with our conventions, as can be seen *e.g.* from the fact that the components of the total super stress tensor computed from the (anti-)commutator of  $Q_B$  with  $b(z)$ ,  $\beta(z)$  and  $\bar{b}(\bar{z})$  satisfy the correct operator product relations. Finally the picture changing operator  $\mathcal{X}$  is given by [32, 33]

$$\mathcal{X}(z) = \{Q_B, \xi(z)\} = c\partial\xi + e^\phi T_F - \frac{1}{4}\partial\eta e^{2\phi}b - \frac{1}{4}\partial(\eta e^{2\phi}b). \quad (3.7)$$

This is a BRST invariant dimension zero primary operator and carries picture number 1.

We shall define the subspace  $\mathcal{H}_0$  of off-shell states in the matter ghost conformal field theory as in (2.16) with some additional restrictions:

$$|\Psi\rangle \in \mathcal{H}_0 \quad \text{if} \quad (b_0 - \bar{b}_0)|\Psi\rangle = 0, \quad (L_0 - \bar{L}_0)|\Psi\rangle = 0, \quad \eta_0|\Psi\rangle = 0, \quad \text{picture number}(|\Psi\rangle) = -1. \quad (3.8)$$

The  $\eta_0|\Psi\rangle = 0$  condition tells us that we are working in the small Hilbert space [32]. It will also be useful to define a subspace  $\mathcal{H}_1$  containing off-shell states of ghost number two in the Siegel gauge

$$|\Psi\rangle \in \mathcal{H}_1 \quad \text{if} \quad |\Psi\rangle \in \mathcal{H}_0, \quad (b_0 + \bar{b}_0)|\Psi\rangle = 0, \quad \text{ghost number}(|\Psi\rangle) = 2. \quad (3.9)$$

## 3.2 The integration measure

For simplicity we shall describe the construction of off-shell amplitudes in the heterotic string theory but the generalization to the case of type II strings is straightforward. On a genus  $g$  Riemann surface, in order to get a non-vanishing correlation function the total picture number of all the operators must add up to  $2g - 2$ . The naive guess would be that the construction of the off-shell amplitude would proceed in a manner identical to that in the case of bosonic string theory except that in the construction of the  $p$ -form in the moduli space the surface state  $\langle \Sigma |$  should be replaced by

$$\langle \Sigma | K, \quad (3.10)$$

where  $K$  is the product of  $2g - 2 + n$  picture changing operators. In this case the picture number carried by the states in  $\mathcal{H}_0$  inserted at the punctures and the picture changing operators add up to  $2g - 2$  as required. We can in fact generalize this a bit by writing

$$K = \sum_{\alpha} A^{(\alpha)} \mathcal{X}(z_1^{(\alpha)}) \mathcal{X}(z_2^{(\alpha)}) \cdots \mathcal{X}(z_{2g-2+n}^{(\alpha)}), \quad (3.11)$$

where the sum over  $\alpha$  in (3.11) runs over arbitrary number of values,  $A^{(\alpha)}$  are arbitrary real numbers satisfying  $\sum_{\alpha} A^{(\alpha)} = 1$  and  $z_1^{(\alpha)}, \dots, z_{2g-2+n}^{(\alpha)}$  are the locations of the picture changing operators for the  $\alpha$ -th term. For definiteness we shall assume that the coordinates  $z_i^{(\alpha)}$  lie on  $\Sigma - \cup_a D_a$  and are measured in the fixed  $z$  coordinate system introduced in §2.1. More precisely if a picture changing operator is located on the component  $\sigma_k$  then its location is measured in the coordinate system  $z_k$  on  $\sigma_k$  that we introduced in §2.1.

With this definition most of the identities satisfied by the off-shell bosonic string theory amplitude generalizes to the heterotic string theory provided we continue to impose the conditions (3.8). There is however one caveat. In proving the analog of (2.18) we have to assume that the constants  $A^{(\alpha)}$  as well as the locations  $z_i^{(\alpha)}$  of the picture changing operators remain fixed as we move in  $\widehat{\mathcal{P}}_{g,n}$  using Schiffer variation. Otherwise in (2.19) the tangent vector  $\widehat{V}_i$  acting on the first term on the right hand side will give additional contributions containing derivatives of  $A^{(\alpha)}$  and  $z_i^{(\alpha)}$  with respect to the coordinates on  $\widehat{\mathcal{P}}_{g,n}$ . While it is certainly possible to keep  $A^{(\alpha)}$  and  $z_i^{(\alpha)}$  fixed locally, various global issues may prevent us from keeping them fixed over the entire section in  $\widehat{\mathcal{P}}_{g,n}$  over which we integrate. Thus we need to allow  $A^{(\alpha)}$  and / or  $z_i^{(\alpha)}$  to depend on the coordinates of  $\widehat{\mathcal{P}}_{g,n}$ . For simplicity we shall take the  $A^{(\alpha)}$ 's to be fixed moduli independent constants and allow the  $z_i^{(\alpha)}$ 's to be moduli dependent – this is not necessary but will suffice for our analysis. A minimal remedy will then be to include extra terms in  $\Omega_p^{(g,n)}$  that can account for moduli dependence of the  $z_i^{(\alpha)}$ 's [21, 33]. However we shall develop a slightly more general formalism that will be useful for dealing with the spurious poles in §4.

This general formalism involves extending  $\widehat{\mathcal{P}}_{g,n}$  to a larger space  $\widetilde{\mathcal{P}}_{g,n}$  by appending the data on the locations  $(z_1, \dots, z_{2g-2+n})$  of  $(2g-2+n)$  picture changing operators to  $\widehat{\mathcal{P}}_{g,n}$ . Thus  $\widetilde{\mathcal{P}}_{g,n}$  can be regarded as a fiber bundle over the base  $\widehat{\mathcal{P}}_{g,n}$ , with  $z_i$ 's acting as fiber coordinates. The tangent vectors of  $\widetilde{\mathcal{P}}_{g,n}$  can be labelled by

1. Schiffer variations generated by  $n$ -tuple of vector fields  $(v^{(1)}(z), \dots, v^{(n)}(z))$  keeping the locations of the picture changing operators fixed in the  $z$  coordinate system,
2.  $\partial/\partial z_i$  for every  $i$ . These move the picture changing operators keeping fixed the Riemann surface, the punctures and the coordinate system on the Riemann surface.

A general choice of picture changing operators like the one given in (3.11) with moduli independent  $A^{(\alpha)}$  can be regarded as a weighted average of several sections of  $\widetilde{\mathcal{P}}_{g,n}$  labelled by  $\alpha$ . Since eventually we shall be interested in integrating forms over these sections, the integral of a form over such weighted averages can be interpreted as the weighted average of the integrals over different sections.

We now define operator valued  $r$ -forms  $K^{(r)}$  along the fiber as follows:

$$K^{(0)} = \mathcal{X}(z_1)\mathcal{X}(z_2)\cdots\mathcal{X}(z_{2g-2+n}), \quad (3.12)$$

$$K^{(r)} = \left[ (\mathcal{X}(z_1) - \partial\xi(z_1)dz_1) \wedge (\mathcal{X}(z_2) - \partial\xi(z_2)dz_2) \wedge \cdots \right.$$

$$\wedge \left( \mathcal{X}(z_{2g-2+n}) - \partial \xi(z_{2g-2+n}) dz_{2g-2+n} \right) \Big]^{(r)}, \quad (3.13)$$

where the superscript  $(r)$  on the right hand side indicates that we need to pick the  $r$  form from the expansion of the terms inside the square bracket. More explicitly, we have

$$\begin{aligned} K^{(1)} &= \sum_{i=1}^{2g-2+n} S_i \prod_{\substack{j=1 \\ j \neq i}}^{2g-2+n} \mathcal{X}(z_j), \\ K^{(2)} &= \sum_{\substack{i,j=1 \\ i < j}}^{2g-2+n} S_i \wedge S_j \prod_{\substack{k=1 \\ k \neq i,j}}^{2g-2+n} \mathcal{X}(z_k), \\ \dots &= \dots \\ K^{(2g-2+n)} &= S_1 \wedge S_2 \wedge \dots \wedge S_{2g-2+n}, \end{aligned} \quad (3.14)$$

where

$$S_i \equiv -dz_i \partial \xi(z_i). \quad (3.15)$$

$K^{(r)}$ 's satisfy the 'descent relations' [21]

$$d_F K^{(r)} = (-1)^{r+1} [K^{(r+1)}, Q_B] \quad \text{for } 1 \leq r \leq 2g-2+n, \quad d_F K^{(2g-2+n)} = 0, \quad (3.16)$$

where  $d_F$  denotes exterior derivative along the fiber direction labelled by  $\{z_i\}$ :

$$d_F K^{(r)} \equiv \sum_i dz_i \wedge \frac{\partial}{\partial z_i} K^{(r)}. \quad (3.17)$$

The symbol  $[ \ ]$  stands for commutator if  $K^{(r+1)}$  is grassmann even and anti-commutator if  $K^{(r+1)}$  is grassmann odd.

Next we define

$$\Omega_p^{(g,n)}(|\Phi\rangle) = (2\pi i)^{-(3g-3+n)} \langle \Sigma | \mathcal{B}_p | \Phi \rangle, \quad \mathcal{B}_p \equiv \sum_{\substack{r=0 \\ r \leq 2g-2+n}}^p K^{(r)} \wedge B_{p-r}, \quad (3.18)$$

where  $B_p$  has been defined in (2.12). Since the notation is somewhat abstract, we shall now clarify the meaning of (3.18). Let us consider  $p$  tangent vectors  $\{V_1 + U_1, \dots, V_p + U_p\}$  of  $\tilde{\mathcal{P}}_{g,n}$  where each  $V_k$  is associated with a Schiffer variation generated by the  $n$ -tuple of vector fields  $\vec{v}_k(z)$  keeping the coordinates  $z_i$ 's fixed and each  $U_k$  is a vector field of the form  $\sum_i u_{k,i} \partial / \partial z_i$  that generates shift of the location of the picture changing operators keeping the moduli of

the Riemann surface, the punctures as well as the coordinate system on the Riemann surface fixed. Then

$$\begin{aligned} \Omega_p^{(g,n)}(|\Phi\rangle)[V_1 + U_1, \dots V_p + U_p] &= (2\pi i)^{-(3g-3+n)} \left\langle \Sigma \left| \sum_{\substack{r=0 \\ r \leq 2g-2+n}}^p \sum_{\mathcal{G}_r} (-1)^{\mathbf{P}} K^{(r)}[\{U_i; i \in \mathcal{G}_r\}] \right. \right. \\ &\quad \left. \left. \wedge B_{p-r}[\{V_j; j \in \mathcal{G}_r^c\}] |\Phi\rangle \right. \right\rangle, \end{aligned} \quad (3.19)$$

where the sum over  $\mathcal{G}_r$  runs over all subsets of length  $r$  of  $1, \dots, p$ ,  $\mathcal{G}_r^c$  denotes the complement of  $\mathcal{G}_r$  and  $(-1)^{\mathbf{P}}$  is the sign that is picked up while rearranging  $1, \dots, p$  to the arrangement  $\{i \in \mathcal{G}_r\}, \{j \in \mathcal{G}_r^c\}$ .  $[\ ]$  on the left hand side denotes contraction with the arguments inside the square bracket as usual.

In words these rules may be stated as follows.

1.  $\Omega_0^{(g,n)}(|\Phi\rangle)$  is given by the correlation function of the vertex operators describing the state  $|\Phi\rangle$ , inserted using the local coordinate system associated with the point in  $\tilde{\mathcal{P}}_{g,n}$  we are at, with additional insertion of  $K^{(0)}$  into the correlation function.
2.  $\Omega_p^{(g,n)}$  is defined in terms of  $\Omega_0^{(g,n)}$  as follows. For every contraction with a tangent vector  $\partial/\partial z_i$  we replace the  $\mathcal{X}(z_i)$  term by  $-\partial\xi(z_i)$ . On the other hand for every contraction of  $\Omega_p^{(g,n)}$  with a tangent vector of  $\tilde{\mathcal{P}}_{g,n}$  associated with Schiffer variation by the  $n$ -tuple of vector fields  $\vec{v}$ , we insert into the correlation function a  $b(\vec{v})$ .

Generalization of the ghost number conservation equation (2.15) tells us that if  $|\Phi\rangle$  has total ghost number  $n_\Phi$  and total picture number  $p_\Phi$  then for  $\Omega_p^{(g,n)}$  to be non-zero we must have

$$n_\Phi - p = 6 - 6g, \quad p_\Phi = -n. \quad (3.20)$$

The second condition is automatically satisfied if  $|\Phi\rangle \in \mathcal{H}_0$  with  $\mathcal{H}_0$  defined as in (3.8).

### 3.3 Properties of the integration measure

First we shall verify that the property mentioned below (2.15) and the two properties mentioned below (2.17) hold once we restrict the string states to satisfy (3.8). The proof of the properties mentioned below (2.17) are identical to that in the case of bosonic string theory. This justifies our regarding  $\Omega_p^{(g,n)}$  as a form in  $\tilde{\mathcal{P}}_{g,n}$  rather than in a larger space that we get by appending the picture changing data to  $\mathcal{P}_{g,n}$ . The proof of the property mentioned below (2.15) however is



more subtle. Let  $v(z)$  denote a globally defined vector field on  $\Sigma - \cup_a D_a$  and  $\vec{v} = (v^{(1)}, \dots, v^{(n)})$  denote the collection of vectors obtained from the restriction of  $v(z)$  on  $\partial D_a$ . This generates a deformation of the local coordinates around the punctures according to (2.3)-(2.6), but this can be undone by a change in the coordinate system  $z$  in  $\Sigma - \cup_a D_a$  with  $z + \epsilon v(z)$  as the new coordinate. This is the reason why earlier this generated a vanishing tangent vector of  $\mathcal{P}_{g,n}$  and hence also of  $\widehat{\mathcal{P}}_{g,n}$ . But in superstring theory, the change in the  $z$  coordinates, required to undo the deformation of the local coordinates, will move the location  $z_i$  of the picture changing operators by  $\epsilon v(z_i)$ . Thus we expect that the tangent vector of  $\widetilde{\mathcal{P}}_{g,n}$  associated with the vector field  $v$  will not vanish but should be equal to

$$U = \sum_i v(z_i) \frac{\partial}{\partial z_i}. \quad (3.21)$$

Equivalently  $V - U$  should vanish as a tangent vector of  $\widetilde{\mathcal{P}}_{g,n}$ . We shall now try to verify that the contraction of  $\Omega_p^{(g,n)}$  with such a tangent vector vanishes. Let  $V$  denote the tangent vector of  $\widetilde{\mathcal{P}}_{g,n}$  associated with the vector field  $v(z)$ . Denoting this contraction of  $\Omega_p^{(g,n)}$  with  $V$  by  $\Omega_p^{(g,n)}[V]$  and using (3.18) we get

$$\Omega_p^{(g,n)}[V] = (2\pi i)^{-(3g-3+n)} \langle \Sigma | \sum_{r=0}^{2g-2+n} (-1)^r K^{(r)} \wedge B_{p-r}[v] | \Phi \rangle. \quad (3.22)$$

where

$$B_{p-r}[v] \equiv b(v) B_{p-r-1}, \quad b(v) \equiv \oint v(z) b(z) dz + \oint \bar{v}(\bar{z}) \bar{b}(\bar{z}) d\bar{z}. \quad (3.23)$$

We can now move  $b(v)$  to the left of the  $K^{(r)}$  using the expression (3.13) for the  $K^{(r)}$ 's and the anti-commutation relations:

$$[\mathcal{X}(z), b(v)] = -v(z) \partial \xi(z), \quad \{S_i, b(v)\} = 0. \quad (3.24)$$

Once  $b(v)$  moves to the left we can contract the integration contour by deforming it into  $\Sigma$  and the contribution vanishes. Thus the net contribution to the right hand side of (3.22) comes from the commutators and can be expressed as

$$-(2\pi i)^{-(3g-3+n)} \left\langle \Sigma \left| \sum_{\ell=1}^{2g-2+n} v(z_\ell) \partial \xi(z_\ell) \sum_{r=0}^{2g-2+n} \sum_{\substack{i_1, \dots, i_r=1 \\ i_1 < i_2 < \dots < i_r, i_s \neq \ell}}^{2g-2+n} S_{i_1} \wedge \dots \wedge S_{i_r} \prod_{\substack{k=1 \\ k \neq \ell, i_1, \dots, i_r}}^{2g-2+n} \mathcal{X}(z_k) B_{p-r-1} \right| \Phi \right\rangle. \quad (3.25)$$

Using (3.14) this can be interpreted as

$$(2\pi i)^{-(3g-3+n)} \left\langle \Sigma \left| \sum_{r=0}^{2g-2+n} K^{(r+1)}[U] \wedge B_{p-r-1} \right| \Phi \right\rangle = \Omega_p^{(g,n)}[U] \quad (3.26)$$

where  $U$  has been defined in (3.21). This shows that  $\Omega_p^{(g,n)}[U] = \Omega_p^{(g,n)}[V]$ , precisely as expected.

Next we shall show that  $\Omega_p^{(g,n)}$  satisfies the identity (2.18) with  $d$  now denoting the exterior derivative in  $\tilde{\mathcal{P}}_{g,n}$ . For this we write

$$\begin{aligned} (2\pi i)^{3g-3+n} \Omega_p^{(g,n)}(Q_B|\Phi) &= \left\langle \Sigma \left| \sum_{r=0}^{2g-2+n} K^{(r)} \wedge [B_{p-r}, Q_B] \right| \Phi \right\rangle \\ &\quad + \left\langle \Sigma \left| \sum_{r=0}^{2g-2+n} (-1)^{p-r} [K^{(r)}, Q_B] \wedge B_{p-r} \right| \Phi \right\rangle, \end{aligned} \quad (3.27)$$

using the relation  $\langle \Sigma | Q_B = 0$ . Now using the same manipulations as in the case of bosonic string theory, the first term on the right hand side can be interpreted as

$$\sum_{r=0}^{2g-2+n} (-1)^{p-r} (-1)^r d_T \langle \Sigma | K^{(r)} \wedge B_{p-r-1} | \Phi \rangle, \quad (3.28)$$

where  $d_T \equiv d - d_F$  in the ‘tangential exterior derivative along the base  $\hat{\mathcal{P}}_{g,n}$ ’ defined so that its contraction with a tangent associated with Schiffer variation *keeping  $z_i$ ’s fixed* is given as in (2.19) while its contraction with  $\partial/\partial z_i$  vanishes.<sup>2</sup> The  $(-1)^{p-r}$  in (3.28) is the result of the  $(-1)^p$  factor in (2.18), while the  $(-1)^r$  factor is the result of passing  $d_T$  through the  $r$ -form  $K^{(r)}$ . On the other hand, using (3.16) we can express the second term on the right hand side of (3.27) as

$$\sum_{r=0}^{2g-2+n} (-1)^{p-r} (-1)^r d_F \langle \Sigma | K^{(r-1)} \wedge B_{p-r} | \Phi \rangle = (-1)^p \sum_{r=0}^{2g-2+n} d_F \langle \Sigma | K^{(r)} \wedge B_{p-r-1} | \Phi \rangle. \quad (3.29)$$

In going from the left hand side to the right hand side of this equation we have made an  $r \rightarrow r+1$  shift and used that the  $r=0$  term on the left hand side and  $r=2g-2+n$  terms on the right hand side vanishes. Adding (3.28) and (3.29) and using the fact that  $d_F + d_T$  represents the total exterior derivative  $d$  on  $\tilde{\mathcal{P}}_{g,n}$ , we arrive at the equation

$$\Omega_p^{(g,n)}(Q_B|\Phi) = (-1)^p d \Omega_{p-1}^{(g,n)}(|\Phi\rangle). \quad (3.30)$$

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<sup>2</sup>Neither  $d_T$  nor  $d_F$  are good operators in the sense that neither of them satisfies the constraint that their contraction with  $V$  associated with Schiffer variation by a vector field  $v(z)$  that is globally defined on  $\Sigma - \cup_a D_a$ , and  $U$  given in (3.21), are equal. But  $d_T + d_F$ , which is the full exterior derivative operator in  $\tilde{\mathcal{P}}_{g,n}$ , has this property.

### 3.4 General parametrization of $\tilde{\mathcal{P}}_{g,n}$

We can also generalize the above prescription to the more general labelling of the tangent vectors of  $\widehat{\mathcal{P}}_{g,n}$  as described in §2.6. In this formalism we describe a Riemann surface as different components  $\sigma_k$  and the coordinate disks  $D_a$ , each with its own coordinate system, glued together at their boundary circles. We can move in  $\widehat{\mathcal{P}}_{g,n}$  by changing the functional relationship between the coordinates of two components sharing a common boundary circle. Thus a particular deformation of one of these functions, encoded in a vector field defined around the common boundary, will describe a tangent vector of  $\widehat{\mathcal{P}}_{g,n}$ . The contraction of the  $p$ -form  $\Omega_p^{(g,n)}$  in bosonic string theory with such a tangent vector involves inserting into the correlator contour integrals of  $b$  and  $\bar{b}$  along the common boundary circles of two components, weighted by the vector field  $v(z)$  and  $\bar{v}(\bar{z})$  associated with this deformation.

In heterotic or type II string theory, we need to insert picture changing operators in the correlation function to get a proper integration measure. As already described earlier, if a particular picture changing operator is located on the component  $\sigma_k$  then its coordinate  $z_i$  is measured in the  $z_k$  coordinate system. Then  $\tilde{\mathcal{P}}_{g,n}$  can be constructed as a fiber bundle over  $\widehat{\mathcal{P}}_{g,n}$  with  $z_i$ 's as the fiber coordinates. The tangent vectors along the fiber are linear combinations of  $\partial/\partial z_i$ 's and generate shift of the locations of the picture changing operators, keeping fixed the Riemann surface, the punctures and the coordinate systems  $\{z_k\}$  and  $\{w_a\}$  on the different components  $\{\sigma_k\}$  and  $\{D_a\}$  of the Riemann surface.  $K^{(r)}$ 's are now defined as in (3.13). The tangent vectors along the base  $\widehat{\mathcal{P}}_{g,n}$ , described in the last paragraph, are lifted to  $\tilde{\mathcal{P}}_{g,n}$  by identifying them as deformations that generate the same deformations along the base  $\widehat{\mathcal{P}}_{g,n}$ , and keeps, for every  $i$ , the coordinate  $z_i$  of the  $i$ -th picture changing operator fixed. With these modifications  $\Omega_p^{(g,n)}$  is defined in the same way as in (3.18), except that, as described in the last paragraph, the  $b(v) = \oint v(z)b(z) + \oint \bar{v}(\bar{z})\bar{b}(\bar{z})$ 's are now more general objects which use vector fields  $v(z)$  describing the changes in the functional relationship between different components and the contour integral runs along the boundary circle separating two such components.

Now it is clear that the location of the boundary between two adjacent components  $\sigma_i$  and  $\sigma_j$  is somewhat arbitrary and can be shifted without changing the relation between  $z_i$  and  $z_j$ . But due to this shift a picture changing operator initially located in  $\sigma_i$  may move to  $\sigma_j$  or vice versa. Since this change is not physical, the integration measure should not change. While the proof of this is straightforward, we shall illustrate this only by an example. Let us consider the case described in (2.25) and pretend that  $q$  is the only modulus of  $\widehat{\mathcal{P}}_{g,n}$  that we

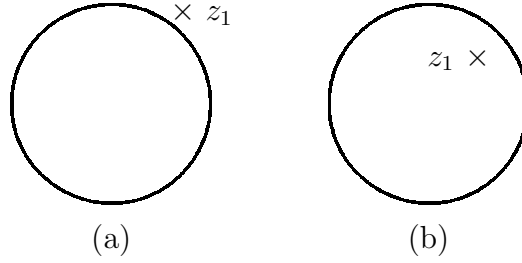


Figure 1: Pictorial representation of the integration contours for eqs.(3.31) and (3.34).

are interested in. Suppose further that we have only one picture changing operator that we want to insert at the point  $z_1$  in the  $z$  coordinate system. Then  $\tilde{\mathcal{P}}_{g,n}$  will be labelled by  $(q, z_1)$  and the 1-form  $\Omega_1^{(g,n)}$  will be associated with the insertion<sup>3</sup>

$$- dq q^{-1} \mathcal{X}(z_1) \oint dz z b(z) - dz_1 \partial \xi(z_1), \quad (3.31)$$

where it is understood that  $z_1$  is placed away from the origin relative to the integration contour in the  $z$ -plane (see Fig. 1(a)). On the other hand if we want to insert the picture changing operator in the  $w$  coordinate system, then  $\Omega_1^{(g,n)}$  is associated with the insertion

$$- dq q^{-1} \mathcal{X}(w_1) \oint dw w b(w) - dw_1 \partial \xi(w_1), \quad (3.32)$$

with  $w_1$  being away from the origin relative to the integration contour in the  $w$  plane.

Let us suppose that in the two cases the picture changing operators are located at the same physical position. This can be achieved by moving the (artificial) boundary between the two components labelled by  $z$  and  $w$  across the location of the picture changing operator. In that case we should get identical results using (3.31) and (3.32). Let us test this. First noting that  $b$  is a primary of dimension 2,  $\partial \xi$  is a primary of dimension 1 and  $\mathcal{X}$  is a primary of dimension 0, we have

$$\mathcal{X}(w) = \mathcal{X}(z), \quad \partial \xi(w) = (\partial z / \partial w) \partial \xi(z) = -q^{-1} z^2 \partial \xi(z), \quad b(w) = (\partial z / \partial w)^2 b(z) = q^{-2} z^4 b(z), \quad (3.33)$$

where the argument of an operator is also used to denote the coordinate system in which it is inserted. Thus (3.32) can be written as

$$- dq q^{-1} \oint dz z b(z) \mathcal{X}(z_1) + q^{-1} z_1^2 dw_1 \partial \xi(z_1). \quad (3.34)$$

---

<sup>3</sup>For  $\Omega_0^{(g,n)}$  and  $\Omega_2^{(g,n)}$  the analysis is trivial.

In writing the above equation we have taken into account the fact that an anti-clockwise contour in  $w$  plane produces a clockwise contour in the  $z$  plane and hence costs an extra  $-$  sign when we make this into an anticlockwise contour  $\oint$ . In the first term we have placed  $\mathcal{X}(z_1)$  on the right of the contour integral, signifying the fact that in (3.34) the point  $z_1$  is towards the origin in the  $z$  plane relative to the integration contour (see Fig. 1(b)) since in the  $w$ -plane it was away from the origin. Using the relation  $w_1 = q/z_1$ , we now write, as a differential form in  $\tilde{\mathcal{P}}_{g,n}$  labelled by  $q$  and  $z_1$ ,

$$dw_1 = -qz_1^{-2}dz_1 + z_1^{-1}dq. \quad (3.35)$$

Substituting this into (3.34) we get

$$-dq q^{-1} \oint dz z b(z) \mathcal{X}(z_1) - dz_1 \partial \xi(z_1) + q^{-1} z_1 dq \partial \xi(z_1), \quad (3.36)$$

where the integration contour is still as shown in Fig. 1(b). In order to compare with (3.31) we now move the contour of integration in the first term through the point  $z_1$  so that  $z_1$  is situated away from the origin relative to the integration contour. In that process we pick up a residue of the form  $-dq q^{-1} z_1 \partial \xi(z_1)$  which precisely cancels the last term in (3.36). Thus we get

$$-dq q^{-1} \mathcal{X}(z_1) \oint dz z b(z) - dz_1 \partial \xi(z_1). \quad (3.37)$$

This precisely agrees with (3.31).

### 3.5 Off-shell amplitude of NS sector fields

For defining off-shell amplitudes in superstring theory we shall need to regard  $\tilde{\mathcal{P}}_{g,n}$  as a fiber bundle over the base  $\mathcal{M}_{g,n}$  and integrate  $\Omega_{6g-6+2n}^{(g,n)}$  on a section – or more generally on the formal weighted average of several sections as in (3.11) – of this fiber bundle. This means that for every point in  $\mathcal{M}_{g,n}$  we need to make a choice of local coordinate system around each puncture and the locations of the picture changing operators, and integrate  $\Omega_{6g-6+2n}^{(g,n)}$  on the subspace of  $\tilde{\mathcal{P}}_{g,n}$  that it defines. The section can be arbitrary subject to the requirement of gluing compatibility. This requires first of all that the choice of local coordinates must be subject to the same kind of constraints as given in §2.7. However we must also put constraint on the  $K^{(r)}$ 's. It requires that when we take a Riemann surface of genus  $g_1 + g_2$  and  $n_1 + n_2 - 2$  punctures, constructed from the plumbing fixture of a Riemann surface of genus  $g_1$  and  $n_1$

punctures with another Reimann surface of genus  $g_2$  and  $n_2$  punctures, then we must have

$$K_{g_1+g_2, n_1+n_2-2}^{(r)} = \sum_{s=0}^r K_{g_1, n_1}^{(s)} \wedge K_{g_2, n_2}^{(r-s)}, \quad (3.38)$$

where  $K_{g,n}^{(s)}$  denotes the choice of  $K^{(s)}$  on the genus  $g$  Riemann surface with  $n$  punctures. Since all the  $K^{(r)}$ 's for  $r \geq 1$  are determined in terms of  $K^{(0)}$ , it is enough to satisfy this equation for  $r = 0$ . In that case it takes the simple form

$$K_{g_1+g_2, n_1+n_2-2}^{(0)} = K_{g_1, n_1}^{(0)} K_{g_2, n_2}^{(0)}. \quad (3.39)$$

This in particular requires that the  $2g-2+n$  picture changing operators on the glued Riemann surface are distributed such that  $2g_1-2+n_1$  of them lie on the first surface and  $2g_2-2+n_2$  of them lie on the second surface [14]. A systematic procedure for constructing such gluing compatible sections will be discussed in §3.7. For such Riemann surfaces we see from (3.38) and (2.29) that after contraction with the tangent vectors of the section,  $\mathcal{B}_{6g-6+2n}$  defined in (3.18) factors as

$$-i \mathcal{B}_{6g_1-6+2n_1}^{(1)} b_0^+ b_0^- \mathcal{B}_{6g_2-6+2n_2}^{(2)}. \quad (3.40)$$

This leads to an exact analog of (2.31), (2.34) for general external states in  $\mathcal{H}_0$  inserted at the punctures:

$$\begin{aligned} \Omega_p^{(g,n)}(|\Phi\rangle)|_S &= \frac{1}{2\pi} \sum_{\substack{0 \leq p_1 \leq 6g_1-6+2n_1, \\ p_1+p_2=p-2}} \sum_{\substack{0 \leq p_2 \leq 6g_2-6+2n_2 \\ i,j}} \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ &\quad (-1)^{p_1 p_2 + N_1 + p_1 + 1} ds \wedge d\theta \wedge \Omega_{p_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{p_2}^{(g_2, n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2}, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \Omega_p^{(g,n)}(|\Phi\rangle)|_{S; s=\Lambda} &= \frac{1}{2\pi} \sum_{\substack{0 \leq p_1 \leq 6g_1-6+2n_1, \\ p_1+p_2=p-1}} \sum_{\substack{0 \leq p_2 \leq 6g_2-6+2n_2 \\ i,j}} \langle \varphi_i^c | b_0^- e^{-\Lambda(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ &\quad (-1)^{p_1 p_2 + p_2} d\theta \wedge \Omega_{p_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{p_2}^{(g_2, n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2}. \end{aligned} \quad (3.42)$$

If we restrict the external states to be in  $\mathcal{H}_1$  and take  $p_i = 6g_i - 6 + 2n_i$  for  $i = 1, 2$  we get

$$\begin{aligned} \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle)|_S &= -\frac{1}{2\pi} \sum_{i,j} ds \wedge d\theta \wedge \Omega_{6g_1-6+2n_1}^{(g_1, n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)|_{S_1} \wedge \Omega_{6g_2-6+2n_2}^{(g_2, n_2)}(|\varphi_j\rangle \times |\Phi_2\rangle)|_{S_2} \\ &\quad \times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle. \end{aligned} \quad (3.43)$$

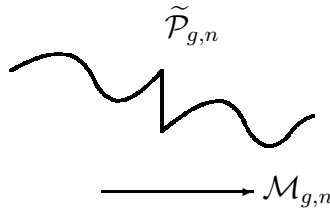


Figure 2: Pictorial representation of a subspace of  $\tilde{\mathcal{P}}_{g,n}$  containing a vertical segment that contains a tangent vector along the fiber.

Since in this case each external state at the punctures carries ghost number two and picture number  $-1$ , it follows from (3.20) that  $|\varphi_i\rangle$ ,  $|\varphi_j\rangle$  carry ghost number two and picture number  $-1$ . Furthermore due to the  $b_0^+ b_0^-$  factor in the second line of (3.43),  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  are annihilated by  $b_0$  and  $\bar{b}_0$ . Thus they correspond to states in  $\mathcal{H}_1$  defined in (3.9).

Note that if we choose the section in such a way that the locations of the picture changing operators remain fixed in the chosen coordinate system on the Riemann surface, then the tangents to the section have vanishing contraction with the  $K^{(r)}$ 's for  $r \geq 1$ . Thus we can set all the  $K^{(r)}$ 's other than  $K^{(0)}$  to zero and get back the usual formalism in which we insert the picture changing operators at fixed points on the Riemann surface. In general however we shall allow the locations of the picture changing operators to vary as we move along the base  $\mathcal{M}_{g,n}$ . We shall also not impose any holomorphicity condition on the section, and allow the locations of the picture changing operators to depend non-holomorphically on the coordinates of  $\mathcal{M}_{g,n}$ .

### 3.6 Vertical integration

If we are only interested in integrating  $\Omega_{6g-6+2n}^{(g,n)}$  over a section, we could from the beginning express this as an integral over  $\mathcal{M}_{g,n}$  by regarding the  $z_i$ 's and the local coordinates as functions of the coordinates of  $\mathcal{M}_{g,n}$ . This will entail replacing the  $dz_i$ 's by  $(\partial z_i / \partial t_k) dt_k$  from the beginning, where  $t_k$  are the coordinates of  $\mathcal{M}_{g,n}$ . However the advantage of regarding the  $\Omega_p^{(g,n)}$ 's as  $p$ -forms on  $\tilde{\mathcal{P}}_{g,n}$  is that we can integrate  $\Omega_p^{(g,n)}$  over any  $p$ -dimensional subspace of  $\tilde{\mathcal{P}}_{g,n}$ . In particular we can use this to carry out ‘vertical integration’ i.e. integration over fibers keeping the base point in  $\mathcal{M}_{g,n}$  fixed. This is necessary for example if we want to choose the integration cycle such that parts of it involves moving the picture changing operators keeping the point in  $\mathcal{M}_{g,n}$  fixed.

A special case of this that will be of interest to us is as follows. Suppose that on a codimension one subspace  $\mathcal{K}$  of  $\mathcal{M}_{g,n}$ , we turn the integration cycle in the vertical direction so that at every point in  $\mathcal{K}$  we change the location  $z_i$  of a particular picture changing operator from its initial value  $u$  to some final value  $v$  keeping fixed the local coordinates at the punctures and the locations of the other picture changing operators. This has been depicted in Fig. 2. Both  $u$  and  $v$ , as well as the local coordinates at the punctures and the locations of other picture changing operators can of course vary along  $\mathcal{K}$ . In this case we can label the vertical part of the integration cycle by the coordinates of  $\mathcal{K}$  and the coordinate  $z_i$  along the fiber, and we can integrate  $\Omega_{6g-6+2n}^{(g,n)}$  over this part on the integration cycle. This integration can be performed by first integrating along the fiber labelled by  $z_i$  and then integrating the result along  $\mathcal{K}$ . Integration along the fiber will give

$$\int_u^v dz_i \Omega_{6g-6+2n}^{(g,n)} \left[ \frac{\partial}{\partial z_i} \right]. \quad (3.44)$$

Now it follows from (3.13)-(3.19) that contraction with  $\partial/\partial z_i$  effectively replaces the  $\mathcal{X}(z_i) - \partial\xi(z_i)dz_i$  term in (3.13) by  $-\partial\xi(z_i)$ . Since the rest of the operators entering the definition of  $\Omega_p^{(g,n)}$  have no dependence on  $z_i$ , the integration along the fiber direction produces an insertion of

$$- \int_u^v dz_i \partial\xi(z_i) = (\xi(u) - \xi(v)), \quad (3.45)$$

into the correlation function. This replaces the  $\mathcal{X}(z_i) - \partial\xi(z_i)dz_i$  term in (3.13). Note in particular that the result depends only on the initial and final values of  $z_i$  and not on the contour in the  $z_i$ -plane along which we integrate. At the same time it must be noted that this operator belongs to the small Hilbert space since it represents the difference in  $\xi$  at two points. The rest of the insertions into the correlation function can be determined from the contraction of  $\Omega_{6g-7+2n}^{(g,n)}$  with the tangent vectors of the image of  $\mathcal{K}$  in  $\tilde{\mathcal{P}}_{g,n}$ .

Note that the above result is valid only if we vary the location of only one picture changing operator. If we want to use the above result to move the locations of several picture changing operators then this can be done by making the vertical segment composed of several parts, and in each part we vary the location of only one picture changing operator.<sup>4</sup> The result will depend on the order in which we vary the locations of the picture changing operators, but this can be addressed in the same way as the effect of a general variation in the choice of integration cycle as described in §3.9.

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<sup>4</sup>If this breaks some symmetry *e.g.* permutation symmetry among external punctures, we can always average over various possibilities. This will increase the number of  $A^{(\alpha)}$ 's.



For reasons that will become clear in §4, we need to define the superstring amplitude by integration over integration cycles containing such vertical segments. In all subsequent discussion we shall be working with such general choice of integration cycle.

### 3.7 Construction of gluing compatible integration cycles

An algorithm for constructing gluing compatible sections in bosonic string theory has been described in §3.2, §3.3 of [10]. We can follow a similar procedure for constructing gluing compatible integration cycles in superstring theory and use it to divide the contributions to a given off-shell amplitude into one particle irreducible (1PI) and one particle reducible (1PR) parts. First of all on three punctured spheres and all one punctured tori parametrized by the torus modular parameter  $\tau$  we make some specific choice of the local coordinates at the punctures subject to the symmetry that permutes the three punctures. A class of choices can be found in [12,42,43]. We also fix the location of the picture changing operator consistent with this symmetry – this may require averaging over more than one set of choices as represented by the sum over  $\alpha$  in (3.11). Also for reasons to be clear later we shall take the locations of the picture changing operators to be in the region  $|w_a| > 1$  for each  $a$  where  $w_a$  denotes the local coordinate around the  $a$ -th puncture. This of course presupposes that the  $|w_a| < 1$  regions are sufficiently small so that they do not cover the whole surface but this can always be done by scaling the  $w_a$ 's by a sufficiently small number  $\lambda$ , and we shall only work with such choices of local coordinates. We declare the three punctured sphere and all one punctured tori to be 1PI Riemann surfaces. We can now glue pairs of these 1PI surfaces using the plumbing fixture relations to construct four punctured spheres and two punctured tori<sup>5</sup> and choose the local coordinates at the punctures and the locations of the picture changing operators to be those induced from the original Riemann surfaces which are being glued. Since the picture changing operators were taken to be in the region  $|w_a| > 1$  for each puncture including the ones that are being used for gluing, it follows that after gluing they are at distinct points on the final Riemann surface for all  $s$  and  $\theta$  as long as we take  $s \geq 0$ . During this construction we treat the external punctures as distinct so that *e.g.* while gluing two three punctured spheres to get a four punctured sphere we get separate contributions from  $s$ ,  $t$  and  $u$  channel diagrams. We call the family of Riemann surfaces obtained this way 1PR Riemann surfaces and declare the rest of the four punctured spheres and two punctured tori as 1PI Riemann surfaces. We

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<sup>5</sup>We can also construct zero punctured genus two Riemann surfaces by gluing a pair of one punctured tori, but these play no role in our analysis.

now make some choice of local coordinates and picture changing operators (possibly including vertical segments) on the new 1PI Riemann surfaces consistent with permutation symmetry, the  $|w_a| > 1$  conditions for the locations of the picture changing operators and the requirement that both the choice of local coordinates and the locations of the picture changing operators smoothly match onto those on 1PR Riemann surfaces on the codimension one subspace of the moduli space that forms the common boundary of the moduli space of 1PI Riemann surfaces and the moduli space of 1PR Riemann surfaces. We continue this process by constructing, for any given value of  $g$  and  $n$ , all 1PR Riemann surfaces by gluing two or more 1PI Riemann surfaces with lower  $g$  and / or  $n$ . At each stage the Riemann surfaces which cannot be obtained by gluing two or more 1PI surfaces of lower genera / lower number of punctures are declared to be 1PI.

Once the division into 1PI and 1PR Riemann surfaces have been made, the 1PI off-shell amplitudes are defined by restricting the integral of  $\Omega_{6g-6+2n}^{(g,n)}$  to run over 1PI Riemann surfaces only. The rest of the contributions to the amplitude are defined to be 1PR. Clearly this division depends on the choice of local coordinates on the 1PI surfaces, but the analysis of [9, 10] shows that the physical renormalized masses and S-matrix elements are independent of the choice of local coordinates.<sup>6</sup>

### 3.8 Infrared regulator

The off-shell amplitudes can have infrared divergences from separating type degenerations represented by the  $s \rightarrow \infty$  limit of (2.28). These can be divided into two kinds – generic degenerations where the momentum flowing through the punctures being glued is general off-shell momentum and special degenerations where the momentum flowing through the punctures being glued is forced to be zero or on-shell [44]. In the case of generic degenerations – which also include all non-separating type degenerations – we regulate the divergence as  $s \rightarrow \infty$  by making an analytic continuation  $s \rightarrow i s$  and include a damping factor  $e^{-\epsilon s}$  in the integral [44]. On the other hand for special degeneration we restrict the  $s$  integral by some upper cut-off  $\Lambda$  [14].

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<sup>6</sup>For some choice of local coordinates on 1PI surfaces it may happen that a given 1PR Riemann surface may appear more than once as a result of gluing *e.g.* once from  $s$ -channel diagram and once from  $t$ -channel diagram in the case of four punctured sphere. In that case the definition of 1PI family of surfaces will require us to subtract this contribution. This can be avoided by scaling the choice of local coordinates of the original 1PI surfaces by some small number  $\lambda$  which reduces the size of the moduli space covered by the 1PR family of Riemann surfaces. A definite choice of local coordinate system that avoids this can be found in [12].

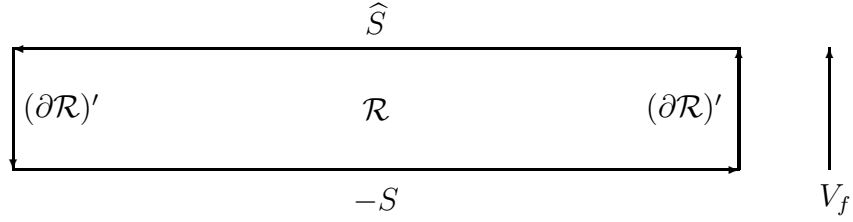


Figure 3: Illustration of (3.46) and (3.49)

Degenerations which glue two Riemann surfaces each of which carries two or more external punctures are always treated as generic. Degenerations which glue two Riemann surfaces of which one has only one external puncture (other than the puncture that is being glued) correspond to mass and wave-function renormalization. If the mass is renormalized then we'll have to work with off-shell amplitudes where we take the external momentum carried by the particle to be at a generic off-shell value [9, 10] and hence we are forced to treat this as a generic degeneration. On the other hand if the mass is not renormalized then we can keep the momentum carried by the state on-shell and treat this as a special degeneration. However we can also work with off-shell external momentum and treat this as a generic degeneration and take the momentum on-shell at the end of the computation. Presumably both methods will lead to the same result for the wave-function renormalization factor at the end but a formal proof of this has not been given. In any case since the wave-function renormalization factor is not a physical observable and, in particular, depends on the choice of local coordinates, this is not a pressing issue.

Separating type degenerations in which one of the Riemann surfaces has no external puncture and the other carries all external punctures are always treated as special, since the momentum flowing through the punctures that are glued is forced to be zero.

### 3.9 Effect of changing the locations of the picture changing operators

Once we have chosen a gluing compatible integration cycle, we need to follow the procedure of [9, 10] to show that even though the off-shell amplitudes depend on the choice of local coordinate system at the punctures, physical quantities like the renormalized mass or the S-matrix elements are independent of this choice. A new question that arises for superstrings is: how does the amplitude depend on the choice of the locations of picture changing operators? We can address this question together with the old question: how does the amplitude depend

on the choice of local coordinates? Both correspond to a change in the integration cycle of  $\tilde{\mathcal{P}}_{g,n}$  on which we integrate to find the amplitude. If  $S$  and  $\hat{S}$  are two integration cycles and  $R$  is the region bounded by them then we have (see Fig. 3)

$$\partial R = \hat{S} - S + (\partial R)', \quad (3.46)$$

where  $(\partial R)'$  is the intersection of  $R$  with the boundary of  $\tilde{\mathcal{P}}_{g,n}$  containing degenerate Riemann surfaces. This gives

$$\int_{\hat{S}} \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle) - \int_S \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle) = \int_R d\Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle) - \int_{(\partial R)'} \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle). \quad (3.47)$$

Using (3.30) we can express the right hand side of (3.47) as

$$- \int_R \Omega_{6g-5+2n}^{(g,n)}(Q_B|\Phi\rangle) - \int_{(\partial R)'} \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle). \quad (3.48)$$

It is useful to write down the result for infinitesimal change which can be parametrized by some infinitesimal vector  $V_f$  of the tangent space of  $\mathcal{P}_{g,n}$  at every point on the original integration cycle  $S$ , labelling the displacement between  $S$  and  $\hat{S}$ . Clearly  $V_f$  is defined only up to the addition of a tangent vector of  $S$ , but this will not affect the final result. In that case (3.48) takes the form

$$- \int_S \Omega_{6g-5+2n}^{(g,n)}(Q_B|\Phi\rangle)[V_f] + \int_{\partial S} \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle)[V_f], \quad (3.49)$$

where  $\partial S$  denotes the intersection of  $S$  with the boundary of  $\tilde{\mathcal{P}}_{g,n}$  containing degenerate Riemann surfaces.

From now on we shall focus on the effect of changing the locations of the picture changing operators, but the arguments given below can be easily generalized to give an alternative analysis of the effect of the change in the local coordinate system, leading to the same results as in [9, 10]. First let us consider the first term in (3.49). For a gluing compatible choice of coordinate system, we can follow the procedure reviewed in §3.7 to break up the integral over  $S$  as sum of 1PI contributions, two 1PI contributions joined by a propagator, three 1PI contributions joined by two propagators etc. It will be useful for our analysis to express the 1PR contributions in terms of the constituent 1PI contributions. The basic identity that allows us to do this can be derived by analyzing a region of the moduli space where the Riemann surface  $\Sigma$  is constructed by gluing two Riemann surfaces  $\Sigma_1$  and  $\Sigma_2$  by plumbing fixture. In this case  $V_f$  can be expressed as  $V_f^{(1)} + V_f^{(2)}$  where  $V_f^{(1)}$  captures the effect of changing the

locations of the picture changing operators on  $\Sigma_1$  and  $V_f^{(2)}$  denotes the effect of changing the locations of the picture changing operators on  $\Sigma_2$ . In this case using a slight generalization of (3.41) to include contraction with the vector  $V_f$  field and that

$$Q_B(|\Phi_1\rangle \otimes |\Phi_2\rangle) = (Q_B|\Phi_1\rangle) \otimes |\Phi_2\rangle + |\Phi_1\rangle \otimes Q_B|\Phi_2\rangle, \quad (3.50)$$

where  $|\Phi_1\rangle$  and  $|\Phi_2\rangle$  denote states inserted at the external punctures of  $\Sigma_1$  and  $\Sigma_2$  respectively, one can show that when restricted to the integration cycle  $S$ , we have

$$\begin{aligned} & \Omega_{6g-5+2n}^{(g,n)}(Q_B|\Phi\rangle)[V_f]|_S \\ = & -\frac{1}{2\pi} \left[ \sum_{i,j} \Omega_{6g_1-5+2n_1}^{(g_1,n_1)}(Q_B|\Phi_1\rangle \otimes |\varphi_i\rangle)[V_f^{(1)}]_{S_1} \wedge \Omega_{6g_2-6+2n_2}^{(g_2,n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)_{S_2} \right. \\ & \quad \wedge ds \wedge d\theta \times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ & - \sum_{i,j} \Omega_{6g_1-6+2n_1}^{(g_1,n_1)}(Q_B|\Phi_1\rangle \otimes |\varphi_i\rangle)_{S_1} \wedge \Omega_{6g_2-5+2n_2}^{(g_2,n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)[V_f^{(2)}]_{S_2} \\ & \quad \wedge ds \wedge d\theta \times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ & - \sum_{i,j} \Omega_{6g_1-5+2n_1}^{(g_1,n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)[V_f^{(1)}]_{S_1} \wedge \Omega_{6g_2-6+2n_2}^{(g_2,n_2)}(|\varphi_j\rangle \otimes Q_B|\Phi_2\rangle)_{S_2} \\ & \quad \wedge ds \wedge d\theta \times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\ & + \sum_{i,j} \Omega_{6g_1-6+2n_1}^{(g_1,n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)_{S_1} \wedge \Omega_{6g_2-5+2n_2}^{(g_2,n_2)}(|\varphi_j\rangle \otimes Q_B|\Phi_2\rangle)[V_f^{(2)}]_{S_2} \\ & \quad \left. \wedge ds \wedge d\theta \times \langle \varphi_i^c | b_0^+ b_0^- e^{-s(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \right]. \quad (3.51) \end{aligned}$$

If  $\Sigma_1$  and  $\Sigma_2$  are 1PI contributions then the right hand side of (3.51) is already expressed in terms of 1PI contributions. If  $\Sigma_1$  and/or  $\Sigma_2$  are obtained as a result of gluing 1PI Riemann surfaces with lower genus and/or lower number of punctures then we need to again express the right hand side in terms of the amplitudes on these Riemann surfaces by making repeated use of (3.41) and (3.51). At the end we get the result in terms of 1PI amplitudes.

Let us now analyze the contribution from the second term in (3.49) that involves an integral over  $\partial S$ . We shall assume that the integration cycle  $S$  (as well as the deformed integration cycle generated by the vector field  $V_f$ ) has been chosen in a modular invariant fashion so that when we regard the base space  $\mathcal{M}_{g,n}$  as the fundamental domain in the Teichmuller space, the contributions from the apparent boundaries, whose different components are related to each other by modular transformations, cancel. Under this assumption, the relevant component of  $\partial S$  arises from separating type degenerations of the Riemann surfaces [14]. Near  $\partial S$  the

Riemann surface  $\Sigma$  is described as the result of gluing two Riemann surfaces using the plumbing fixture relation (2.28), with the degeneration corresponding to the  $s \rightarrow \infty$  limit. As discussed in §3.8, we can divide these into two kinds – the generic degeneration where the momentum flowing through the degenerating punctures is a generic off-shell momentum and the special degeneration in which the momentum flowing through the node is either zero or satisfies a classical on-shell condition. For a generic degeneration we make an analytic continuation  $s \rightarrow is$  and introduce a damping factor  $e^{-\epsilon s}$  in the integral. As a result the  $s$  integral is convergent and there are no boundary contributions from the  $s \rightarrow \infty$  end due to the damping factor. The same argument can be used to rule out boundary contributions from non-separating type degenerations. On the other hand for special degenerations we shall regulate the infrared divergence by putting a sharp cut-off  $s \leq \Lambda$  on the  $s$  integral and let  $\theta$  run over the full range  $0 \leq \theta < 2\pi$ . This boundary contribution can be computed using a slight generalization of (3.42) and gives

$$\begin{aligned}
& \Omega_{6g-6+2n}^{(g,n)}(|\Phi\rangle)[V_f]|_{S;s=\Lambda} \\
= & -\frac{1}{2\pi} \left[ -d\theta \wedge \sum_{i,j} \Omega_{6g_1-5+2n_1}^{(g_1,n_1)}(|\Phi_1\rangle \otimes |\varphi_i\rangle)[V_f^{(1)}]|_{S_1} \wedge \Omega_{6g_2-6+2n_2}^{(g_2,n_2)}(|\varphi_j\rangle \otimes |\Phi_2\rangle)|_{S_2} \right. \\
& \times \langle \varphi_i^c | b_0^- e^{-\Lambda(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_j^c \rangle \\
& + d\theta \wedge \sum_{i,j} \Omega_{6g_1-6+2n_1}^{(g_1,n_1)}(|\Phi_1\rangle \otimes |\varphi_j\rangle)|_{S_1} \wedge \Omega_{6g_2-5+2n_2}^{(g_2,n_2)}(|\varphi_i\rangle \otimes |\Phi_2\rangle)[V_f^{(2)}]|_{S_2} \\
& \left. \times \langle \varphi_j^c | b_0^- e^{-\Lambda(L_0+\bar{L}_0)} e^{i\theta(L_0-\bar{L}_0)} | \varphi_i^c \rangle \right]. \tag{3.52}
\end{aligned}$$

Note that in the second term in (3.52) we have exchanged the roles of  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  compared to the convention used in (3.51). This will be useful later.

### 3.10 Extension to type II strings

The extension of this formalism to type II string theory is in principle straightforward. Now (3.18) will have to be replaced by

$$\Omega_p^{(g,n)}(|\Phi\rangle) = (2\pi i)^{-(3g-3+n)} \langle \Sigma | \mathcal{B}_p | \Phi \rangle, \quad \mathcal{B}_p \equiv \sum_{\substack{r,s=0 \\ r,s \leq 2g-2+n}}^p K^{(r)} \wedge \bar{K}^{(s)} \wedge B_{p-r-s}, \tag{3.53}$$

where  $\bar{K}^{(s)}$  involving the left-moving (anti-holomorphic) fields is defined in the same way as  $K^{(r)}$ . While choosing an integration cycle, in general the weight factors  $\bar{A}^{(\alpha)}$  and the locations

$\bar{z}_i^{(\alpha)}$  of the anti-holomorphic picture changing operators can be chosen to be independent of their holomorphic counterparts. The rest of the analysis will proceed as in the case of heterotic string theory with the understanding that the vector field describing change of locations of the picture changing operators will now correspond to a vector of the form  $\sum_i \delta z_i \partial / \partial z_i + \sum_i \delta \bar{z}_i \partial / \partial \bar{z}_i$ .

## 4 Dealing with spurious poles

Appearance of the second term on the right hand side of (3.49) shows that when we change the location of the picture changing operators, the integration measure in  $\mathcal{M}_{g,n}$  changes by a total derivative term even for on-shell amplitudes for which  $Q_B|\Phi\rangle = 0$ . This is related to the fact that when we regard the amplitude as the result of an integral over the supermoduli space, we have to make a choice of the integration cycle that determines the even nilpotent part of the bosonic moduli. For example if we have a supermoduli space with one bosonic coordinate  $x$  and a pair of fermionic coordinates  $\xi, \eta$ , then for defining an integral of the form

$$\int dx d\xi d\eta f(x, \xi, \eta), \quad (4.1)$$

we need to pick an ‘integration cycle’

$$x = u + h(u)\xi\eta \quad (4.2)$$

where  $u$  is an ordinary bosonic variable and  $h(u)$  is some arbitrary function labelling the even nilpotent part of  $x$ . We then substitute (4.2) into (4.1) and integrate over  $u, \xi, \eta$ . One finds that after  $\xi, \eta$  integration we are left with an integrand that is a function of  $u$ , but it depends on the function  $h(u)$  through a total derivative term.

The consequence of this ambiguity on integration over the supermoduli space has been discussed extensively in [14, 18, 39, 45–47]. We shall not review this here, but only mention that this makes the computation of the amplitude in fermionic string theory tricky under certain circumstances – namely when the supermoduli space is not holomorphically projected [19]. Resolution of this subtlety has been described in [14] by expressing the amplitude as integral over the supermoduli space. In the formalism involving picture changing operators related subtleties arise in the form of spurious poles [33] – poles in the integrand which depend on the locations of the picture changing operators. Since the dependence on the locations of the picture changing operators is a total derivative in the moduli space, one would naively expect that these singularities are ‘fake’ as their locations can be moved around by adding

total derivative terms in the integrand over the moduli space. Nevertheless we have to find a systematic way of dealing with these singularities. This will be the main goal of this section. In the last paragraph of §4.2 we shall briefly discuss the relation between our approach and the one suggested in [14].

## 4.1 Spurious poles

We shall now briefly review the origin of the spurious poles [33]. They arise from the correlation functions of the  $e^{q\phi}$ ,  $\eta$  and  $\xi$  operators. Regarded as a function of the locations of these operators, these correlation functions have zeroes and poles controlled by the operator product expansions of these fields but also have poles at points where no two operators are coincident. This is seen by explicitly writing down the expression for an arbitrary correlator of these fields on a genus  $g$  Riemann surface. Up to an overall normalization it takes the form [33, 48, 49]

$$\begin{aligned} & \left\langle \prod_{i=1}^{n+1} \xi(x_i) \prod_{j=1}^n \eta(y_j) \prod_{k=1}^m e^{q_k \phi(z_k)} \right\rangle_{\delta} \\ &= \frac{\prod_{j=1}^n \vartheta[\delta](-\vec{y}_j + \sum \vec{x} - \sum \vec{y} + \sum q \vec{z} - 2\vec{\Delta})}{\prod_{i=1}^{n+1} \vartheta[\delta](-\vec{x}_i + \sum \vec{x} - \sum \vec{y} + \sum q \vec{z} - 2\vec{\Delta})} \frac{\prod_{i < i'} E(x_i, x_{i'}) \prod_{j < j'} E(y_j, y_{j'})}{\prod_{i,j} E(x_i, y_j) \prod_{k < \ell} E(z_k, z_{\ell})^{q_k q_{\ell}} \prod_k \sigma(z_k)^{2q_k}}, \\ & \sum_{k=1}^m q_k = 2(g-1). \end{aligned} \quad (4.3)$$

In this equation  $\delta$  stands for the spin structure,  $\vartheta$  denotes the genus  $g$  theta functions,  $E(x, y)$  denotes the prime form,  $\sigma(z)$  is a  $\frac{1}{2}g$  differential representing the conformal anomaly of the ghost system and  $\vec{\Delta}$  is the Riemann class characterizing the divisor of zeroes of the theta function. A detailed explanation of all of these quantities can be found in [33, 50].  $\sum \vec{x}$ ,  $\sum \vec{y}$  and  $\sum q \vec{z}$  denote respectively  $\sum_{i=1}^{n+1} \vec{x}_i$ ,  $\sum_{j=1}^n \vec{y}_j$  and  $\sum_{k=1}^m q_k \vec{z}_k$  with

$$\vec{x} \equiv \int_p^x \vec{\omega}, \quad (4.4)$$

where  $\vec{\omega}$  is a  $g$ -dimensional vector of holomorphic one forms on the Riemann surface and  $p$  is an arbitrary point on the Riemann surface (with dependence on  $p$  compensated by  $p$ -dependence of  $\vec{\Delta}$ ).

Note that on the left hand side of (4.3) we have one more  $\xi$  compared to  $\eta$ . This reflects that the correlation function has been written in the ‘large Hilbert space’ in which on any Riemann surface there is a  $\xi$  zero mode that needs to be soaked up. In all operators that we



use for computing amplitudes – vertex operators of external states, picture changing operators, BRST operators etc. –  $\xi$  always appears in the combination  $\partial\xi$ . As these do not carry any zero mode of  $\xi$ , we need to insert an explicit factors of  $\xi(z_0)$  in the correlation function to soak up the  $\xi$  zero mode. Since the correlation function is independent of  $z_0$ , normally we work in the ‘small Hilbert space’ where we do not display the  $\xi(z_0)$  factor in the correlation function. Indeed if we take the derivative of (4.3) with respect to  $n$  of the  $n+1$   $x_i$ ’s, then the correlation function becomes manifestly independent of the last remaining  $x_i$ . We can then drop this from the argument and get the correlation function in the ‘small Hilbert space’. In our analysis we shall always work in the small Hilbert space. In particular, even though in some expressions we may use the operator  $\xi$  without any derivative, they will always appear in a combination  $\xi(A) - \xi(B)$  so that it is really a shorthand for  $\int_A^B \partial\xi(z)dz$ .

The prime form  $E(x, y)$  has a simple zero at  $x = y$  and various prime forms in (4.3) capture the zeroes and poles associated with the short distance singularities / zeroes in the operator product of the various fields. The zeroes of the  $\prod_{i=1}^{n+1} \vartheta[\delta](-\vec{x}_i + \sum \vec{x} - \sum \vec{y} + \sum q \vec{z} - 2\vec{\Delta})$  factor in the denominator are responsible for the spurious poles. With some effort one can see that when we use (3.1) and (4.3) to compute the correlation functions of  $\beta$  and  $\gamma$  this denominator factor becomes independent of the locations of  $\beta$  and  $\gamma$  [48, 49]. Thus there are no spurious poles as functions of the arguments of  $\beta$  and  $\gamma$  operators. Since the BRST current is constructed from  $\beta$  and  $\gamma$  we see that there are no spurious poles in the argument of the BRST current either. However in the expressions for picture changing operators there are ‘bare’  $\partial\xi$ ,  $\eta$  and  $e^{q\phi}$  factors which cannot be expressed as polynomials of (derivatives of)  $\beta$  and  $\gamma$ , and hence the correlators will have spurious singularities as functions of the locations of the picture changing operators. Physically these spurious singularities can be traced to the fact that the gauge choice for the world-sheet gravitinos, which lead to some particular insertion of picture changing operators on the world-sheet, fail to be a good choice of gauge precisely when the correlator of these operators hits a spurious singularity.

Let us now examine how the spurious poles affect the construction of off-shell amplitudes. Let us suppose that we have made some gluing compatible choice of integration cycle in  $\tilde{\mathcal{P}}_{g,n}$  for defining these amplitudes. This corresponds to specifying the locations of the picture changing operators (and choice of local coordinates at the punctures) as a function of the moduli labelling  $\mathcal{M}_{g,n}$ . At a generic point of  $\mathcal{M}_{g,n}$  the  $\prod_{i=1}^{n+1} \vartheta[\delta](-\vec{x}_i + \sum \vec{x} - \sum \vec{y} + \sum q \vec{z} - 2\vec{\Delta})$  factor in the denominator of (4.3) is not expected to vanish, and hence the integration measure is non-singular at a generic point. However since we need to satisfy one complex equation for this

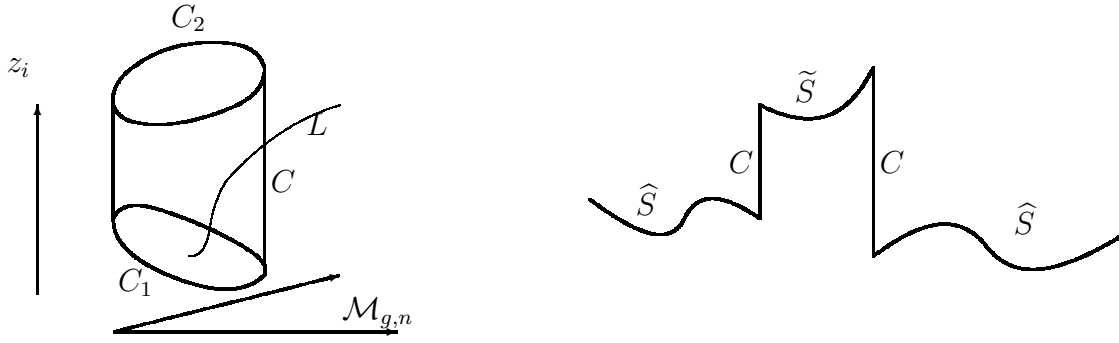


Figure 4: A pictorial representation of integration along a vertical segment. The two horizontal directions in the left hand side figure represent coordinates along  $\mathcal{M}_{g,n}$  transverse to the codimension two subspace on which we have spurious pole and the vertical direction labels the location  $z_i$  of a picture changing operator. All other coordinates of  $\tilde{\mathcal{P}}_{g,n}$  have been suppressed. The integration cycle consists of 3-pieces – a section  $\hat{S}$  (which will be a two dimensional surface in this representation but not shown) whose inner boundary is the curve  $C_1$ , the vertical cylindrical surface  $C$  bounded by  $C_1$  and  $C_2$ , and another section  $\tilde{S}$  (not shown) whose outer boundary is the curve  $C_2$ . The right hand figure shows the intersection of this integration cycle with a vertical plane where we see the three parts  $\hat{S}$ ,  $C$  and  $\tilde{S}$  of the integration cycle explicitly. The thin curve marked  $L$  in the left hand figure describes the location of the spurious pole. As is clear from this figure, both the sections  $\hat{S}$  and  $\tilde{S}$  can avoid the spurious pole. Integration over  $C$  will encounter the spurious pole, but as this is expressed as a difference between an integral along  $C_1$  and an integral along  $C_2$ , this also avoids the spurious pole.

factor to vanish, we expect to encounter spurious poles over a real codimension two subspace of the moduli space.

There are also other more obvious singularities which depend on the locations of the picture changing operators, *e.g.* the ones arising from collision of a pair of picture changing operators or the collision of a picture changing operator with a vertex operator at a puncture. All of these occur on codimension two subspaces of the moduli space and the method we shall describe will deal with all these singularities in the same way.

## 4.2 Dealing with the spurious poles

Our goal will be to describe a procedure for integrating through these spurious singularities. Suppose we have a section of  $\tilde{\mathcal{P}}_{g,n}$  which encounters spurious singularities on a real codimension two subspace  $\mathcal{N}$  of the base  $\mathcal{M}_{g,n}$ . Let us consider a tubular neighborhood  $T$  surrounding  $\mathcal{N}$ . Outside  $T$  the integrand has no spurious poles. Our prescription will be to turn the integration cycle along a ‘vertical direction’ – in the sense described in §3.6 – as we reach the boundary of  $T$  on the base  $\mathcal{M}_{g,n}$ . This corresponds to changing the location  $z_i$  of one of the picture changing operators keeping fixed the locations of the other picture changing operators, local

coordinates around the punctures and the coordinates of  $\mathcal{M}_{g,n}$ . This is done for every point on  $\partial T$ . Thus the vertical segment is a  $6g - 6 + 2n$  dimensional subspace of  $\tilde{\mathcal{P}}_{g,n}$  labelled by the coordinates of  $\partial T$  and the contour along which  $z_i$  varies. The vertical segment is continued till the final arrangement of the picture changing operators as a function of the coordinates on  $\partial T$  are such that there are no longer any spurious poles inside  $T$ . At that point we can turn the section ‘horizontal’ and integrate over the interior of  $T$ . This has been shown pictorially in Fig. 4 where the projection of the curves  $C_1$  and  $C_2$  on  $\mathcal{M}_{g,n}$  correspond to  $\partial T$  and the projection of the interior of the cylindrical region  $C$  on  $\mathcal{M}_{g,n}$  corresponds to  $T$ .

Now naively one might expect this procedure to run into the spurious singularities as we integrate along the vertical direction. After all the location of the spurious poles in  $\mathcal{M}_{g,n}$  must vary continuously as we change the locations of the picture changing operators, and since in the initial configuration there are spurious poles in the interior of  $T$  and in the final configuration there are no spurious poles in the interior of  $T$ , the poles must pass through the boundary of  $T$  as we change the location  $z_i$  of the  $i$ -th picture changing operator. This suggests that the orbit of the spurious pole(s) must cross the vertical segment. This has been shown by the thin line  $L$  in Fig. 4. On the other hand the formalism of §3.6 shows that integrating along the vertical segment, corresponding to integrating  $z_i$  from  $u$  to  $v$  (say), requires us to replace the  $\mathcal{X}(z_i) - \partial\xi(z_i)dz_i$  factor in (3.13) by  $-\int_u^v \partial\xi(z)dz = (\xi(u) - \xi(v))$ . Thus the result depends on only  $u$  and  $v$  and not on the path connecting  $u$  to  $v$ . In particular since for  $\xi(u)$  insertion the spurious poles are in the interior of  $T$  and for  $\xi(v)$  insertion the spurious poles are outside  $T$ , neither the contribution involving  $\xi(u)$  nor the contribution involving  $\xi(v)$  has any singularity on  $\partial T$ . This shows that the result of the vertical integration is free from any spurious singularity.

It is worth examining this in more detail. The point is that if instead of  $\xi(v) - \xi(u)$  we use the expression  $\int_u^v \partial\xi(z)dz$  then somewhere along the contour  $z$  reaches a point where the location of the spurious pole reaches  $\partial T$  causing the integration measure to diverge. However since the correlation function of  $\xi(z)$  given in (4.3) is single valued, the integral  $\int_u^v \partial\xi(z)dz$  can be carried out through this pole unambiguously, leading to correlation function involving  $(\xi(v) - \xi(u))$  which is manifestly free from any poles on  $\partial T$ . This gives a systematic procedure for dealing with spurious poles in the computation of off-shell amplitudes.

This procedure is of course not completely unambiguous since it depends on which  $z_i$  we choose to vary to move the spurious singularity. To see the effect of this, let us compare the results for two different vertical segments of the integration cycle, in each of which the net

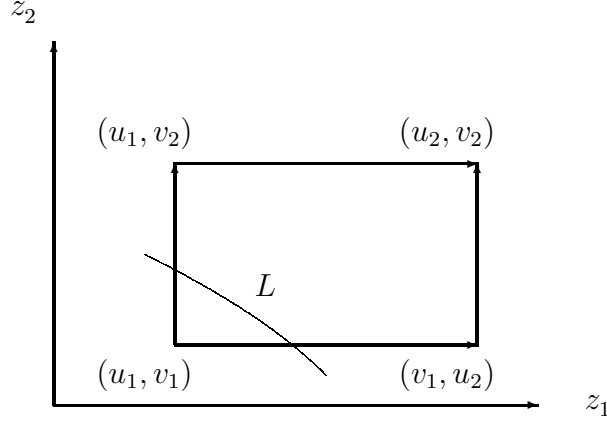


Figure 5: The two integration cycles in the  $z_1$ - $z_2$  plane.

effect is to change the locations of two picture changing operators – which we shall take to be  $z_1$  and  $z_2$  – from  $(u_1, v_1)$  to  $(u_2, v_2)$  in a way that moves the spurious singularity out. In the first case, once we reach a point on  $C_1$ , we first change the location  $z_1$  from  $u_1$  to  $v_1$  to move the spurious singularity out, and then change  $z_2$  from  $u_2$  to  $v_2$ . In the second case we first change  $z_2$  from  $u_2$  to  $v_2$  to move the spurious singularity out, and then change  $z_1$  from  $u_1$  to  $v_1$ . This has been shown in Fig. 5 where we have also displayed the movement of the spurious singularity by the thin line  $L$ . We now calculate the difference between the integral of  $\Omega_{6g-6+2n}^{(g,n)}$  over these two different cycles. Formally this can be expressed as in (3.48) – the only issue is whether the two terms in (3.48) are free from the divergences associated with the spurious poles. First we consider the first term of (3.48). It is clear that this can be obtained by first integrating  $\Omega_{6g-5+2n}^{(g,n)}(Q_B|\Phi)$  over the square in the  $z_1$ - $z_2$  plane shown in Fig. 5 and then integrating the result over the image of  $\partial T$  in  $\tilde{\mathcal{P}}_{g,n}$  sans the directions labelled by  $z_1$  and  $z_2$ . The integral in the  $z_1$ - $z_2$  plane will correspond to the insertion of

$$\int_{u_1}^{v_1} dz_1 \int_{u_2}^{v_2} dz_2 \partial \xi(z_1) \partial \xi(z_2) = (\xi(u_1) - \xi(v_1)) (\xi(u_2) - \xi(v_2)). \quad (4.5)$$

Since there are no spurious singularities at the corner points of the square in the  $z_1$ - $z_2$  plane this gives a finite result. Similarly the second term of (3.48) can be evaluated by first performing the integral of  $\Omega_{6g-6+2n}^{(g,n)}$  over the  $z_1$ - $z_2$  plane, leading to the insertion of (4.5) into the correlation function, and then integrating this over the image in  $\tilde{\mathcal{P}}_{g,n}$  of the intersection of  $\partial T$  with the compactification boundary. The latter arise from setting  $s = \Lambda$  for some plumbing fixture variable  $s$ .

This shows that the difference in the off-shell amplitude for two different choices of vertical segment shown in Fig. 5 can be expressed as (3.48) with finite expressions for both terms. However since (4.5) is not infinitesimal, we do not have the analog of (3.49) which will be needed in §5 to prove that these contributions do not affect renormalized masses and S-matrix elements. So we need to find a way to express this as a result of successive infinitesimal changes. This can be done by taking a family of generalized integration cycles labelled by a continuous parameter  $t$  such that the following conditions hold:

1. For each  $t$  the integration cycle is a formal weighted average of several integration cycles differing in their vertical segments.
2. For each  $t$  the integration cycles satisfy the gluing compatibility condition (3.38).
3. At  $t = 0$  and  $t = 1$  the integration cycles coincide with the original integration cycles which we wanted to show are equivalent.

The effect of infinitesimal change from  $t$  to  $t + \delta t$  will now involve the insertion of terms like (4.5) into the correlation functions as before after carrying out the integral in the  $z_1$ - $z_2$  plane, but the result will be multiplied by a factor of  $\delta t$ . Since this is an infinitesimal deformation, its effect can now be analyzed as in §5 to show that these contributions do not affect the renormalized masses and S-matrix elements.

More generally the rules for choosing picture changing operators will involve dividing up the moduli space into subregions, choose the picture changing operators in each subregion as smooth functions of the moduli ensuring that they do not encounter any spurious singularity, and at the boundary of two such subregions use the prescription of vertical integration to interpolate between the two picture changing operator arrangements in the two subregions. When two or more such boundaries meet there can be additional subtleties since the vertical path across the different boundaries may not be compatible, leaving a ‘gap’ in the integration cycle in  $\tilde{\mathcal{P}}_{g,n}$ .<sup>7</sup> We then have to ‘fill this gap’ by adding new vertical components to the integration cycle in which two of the coordinates are along the ‘vertical direction’ as in (4.5). In unpublished work with E. Witten it has been shown that it is possible to find a consistent procedure for filling these gaps.

One also needs to check that this prescription for dealing with spurious poles is consistent with gluing compatibility. Suppose we have chosen the integration cycles on two families of

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<sup>7</sup>I wish to thank E. Witten for raising this point.

1PI Riemann surfaces avoiding spurious poles. Now consider an 1PR contribution obtained by joining the two families using plumbing fixture. The choice of integration cycles on the original families of Riemann surfaces induces an integration cycle on the family of glued Riemann surfaces. Does this automatically avoid the spurious poles? This can be guaranteed by scaling the local coordinates at the punctures by a sufficiently small number so that 1PR surfaces always describe Riemann surfaces close to degeneration for the whole range  $0 \leq s < \infty$ . In this case the theta function on the glued surface, responsible for the spurious poles, can be approximated by products of theta functions on the original surfaces and hence the locations of the spurious poles will be close to those on the original Riemann surfaces. Thus as long as the original integration cycles were chosen to avoid the spurious poles on the original surface, the induced integration cycle will avoid the spurious poles on the glued surface.

The prescription for avoiding the spurious poles given here may be related to the one suggested in [46], i.e. taking the  $z_i$ 's to be holomorphic function of the moduli around the neighbourhood of the spurious poles and then excluding a small tubular neighbourhood around the spurious poles while carrying out the integral over  $\mathcal{M}_{g,n}$ . However we shall not explore the relation here.

Finally we can also explore the connection between the prescription given here and the suggestion of [14, 51] of performing integration over the supermoduli space by dividing it into open sets in each of which we can choose a good slice for integration over the supermoduli, and then expressing the full result as sum of contributions from such open sets using partition of unity. To see the connection of this to the formalism employed here we note that in Fig. 4 the two sections  $\hat{S}$  and  $\tilde{S}$  can be regarded as two such open sets and the extra term, coming from integration over the vertical segment, can be interpreted as capturing the effect of decreasing the weight given to the choice of section corresponding to  $\hat{S}$  from 1 to 0 across  $\partial T$  and at the same time increasing the weight given to the choice of section corresponding to  $\tilde{S}$  from 0 to 1. Thus the prescription for dealing with the spurious poles, as given here, seems to be in consonance with the one described in [14, 51].

## 5 Mass renormalization and S-matrix of special states

Using the method of [9] one can show that the renormalized masses and S-matrix elements of special states are independent of the choice of local coordinates at the punctures as long as we define the off-shell amplitudes following the procedure described in §3. We shall now

demonstrate how the formalism can be used to prove that the same physical quantities are also independent of the choice of locations of the picture changing operators.<sup>8</sup>

## 5.1 Definition of special states

We shall follow the notation of [9] as closely as possible so that whenever possible we can refer the reader to the analysis of [9]. Let us consider a string theory on  $R^{D,1} \times \mathcal{K}$  where  $\mathcal{K}$  is a compact space and  $R^{D,1}$  is  $D + 1$  dimensional Minkowski space. Then  $SO(D)$  is the little group of a particle at rest. Let  $G$  be the internal symmetry group of the theory (if any). By definition, special states at mass level  $m$  in the rest frame are described by off-shell vertex operators with the following properties:

1. They have the form

$$W_{\pm} = c \bar{c} e^{-\phi} e^{\pm i k_0 X^0} V, \quad (5.1)$$

where  $V$ 's are GSO odd superconformal primary operators of dimension  $(1 + \alpha' m^2/4, 1/2 + \alpha' m^2/4)$  constructed from matter fields and transform in some set of irreducible representations  $R_1, \dots, R_s$  of  $SO(D) \times G$ .

2. *All* zero momentum matter vertex operators in the representation  $R_i$  of  $SO(D) \times G$  for  $1 \leq i \leq s$  have total conformal dimension

$$h \geq \frac{3}{2} + \alpha' \frac{m^2}{2}. \quad (5.2)$$

Furthermore if the equality in (5.2) holds then the operator describes a special state when substituted into (5.1).

The on-shell condition for the vertex operators  $W_{\pm}$  given in (5.1) is  $k^2 = -m^2$ , showing that  $m$  is the tree level mass of the state. We shall denote by  $n_p$  the total number of special states with a given tree level mass  $m$ . In general this may include states from more than one irreducible representation of  $SO(D) \times G$  i.e.  $s$  can be larger than 1. Special states are generalization of states lying on the leading Regge trajectory.

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<sup>8</sup>We should add that the analysis of this and the next section has now been carried out in a much more systematic manner in [52, 53] using the notion of one particle irreducible effective string field theory.

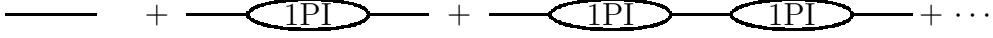


Figure 6: Pictorial representation of the full two point Greens function as a sum of 1PI amplitudes connected by propagators.

## 5.2 Renormalized mass

Now we consider the net contribution to the two point Green's function of two arbitrary off-shell external states  $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}_0$  satisfying the additional restrictions that  $b_0^+|\Psi_i\rangle = 0$  and that they carry momentum  $\pm k = \pm(k^0, \vec{0})$ . Using (3.41) repeatedly, this can be expressed as sum of 1PI contributions glued together by propagators as shown in Fig. 6. This leads to the following expression for the quantum corrected propagator

$$\Pi = \Delta + \Delta \hat{\mathcal{F}} \Delta + \Delta \hat{\mathcal{F}} \Delta \hat{\mathcal{F}} \Delta + \dots = \Delta + \Delta \mathcal{F} \Delta, \quad (5.3)$$

$$\mathcal{F} \equiv \hat{\mathcal{F}} + \hat{\mathcal{F}} \Delta \hat{\mathcal{F}} + \hat{\mathcal{F}} \Delta \hat{\mathcal{F}} \Delta \hat{\mathcal{F}} + \dots = (1 - \hat{\mathcal{F}} \Delta)^{-1} \hat{\mathcal{F}}. \quad (5.4)$$

Here, as in [9],  $\hat{\mathcal{F}}$  is the 1PI contribution to the 2-point amplitude which includes the  $b_0^\pm$  factors appearing in (3.41) and is regarded as an operator acting on states in  $\mathcal{H}_0$  carrying momentum  $k$  and annihilated by  $b_0^+$ .  $\Delta$  is the tree level propagator<sup>9</sup>

$$\Delta = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty ds e^{-s(L_0 + \bar{L}_0) + i\theta(L_0 - \bar{L}_0)} = \delta_{L_0, \bar{L}_0} \int_0^\infty ds e^{-s(L_0 + \bar{L}_0)}. \quad (5.5)$$

$\mathcal{F}$  is the full off-shell two point amplitude. The off-shell two point amplitude of two external states  $|\Psi_1\rangle$  and  $|\Psi_2\rangle$  defined in §3.5, or its 1PI counterpart defined in §3.7, can be obtained by taking the matrix element of  $\frac{1}{2}c_0\bar{c}_0\mathcal{F}$  and  $\frac{1}{2}c_0\bar{c}_0\hat{\mathcal{F}}$  between the states  $\langle\Psi_1|$  and  $|\Psi_2\rangle$  – the  $\frac{1}{2}\bar{c}_0c_0$  factor being needed due to the fact that we have absorbed a  $b_0^+b_0^-$  factor in the definition of  $\mathcal{F}$  and  $\hat{\mathcal{F}}$ . A useful property of the operators  $\mathcal{F}$ ,  $\hat{\mathcal{F}}$  and  $\Delta$  is that they preserve ghost number and picture number, i.e. acting on a state of ghost number  $g$  and picture number  $p$  they give back a state of ghost number  $g$  and picture number  $p$ . They also preserve  $SO(D) \times G$  symmetry, i.e. acting on a state in representation  $R$  of  $SO(D) \times G$ , they give back states in the same representation.

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<sup>9</sup>This definition of  $\Delta$  differs from the definition used in [9, 10] by a normalization factor of 2. This has the effect that in the  $\alpha' = 1$  unit,  $\Delta$  restricted to mass level  $m$  states is given by  $2/(k^2 + m^2)$ . However we can avoid this annoying factor of 2 if we choose  $\alpha' = 2$ .



If  $P$  denotes the projection operator into  $n_p$  dimensional subspace of special states with tree level mass  $m$ , then the off-shell two point Green's function of special states is obtained by multiplying (5.3) by  $P$  from both sides. In  $\alpha' = 2$  unit this is given in terms of  $\mathcal{F}$  by

$$P\Pi(k)P = (k^2 + m^2)^{-1}P + (k^2 + m^2)^{-1}P\mathcal{F}P(k^2 + m^2)^{-1}. \quad (5.6)$$

We can try to evaluate this using (5.4). The poles in  $k^2$  near  $-m^2$  come from the explicit factors of  $(k^2 + m^2)^{-1}$  in (5.6) and the poles of  $\Delta$  in (5.4). The only intermediate states which can generate poles at  $k^2 = -m^2$  from the  $\Delta$  factors in (5.4) are special states themselves since due to (5.2) other states at the same mass level transform in different representations of  $SO(D) \times G$  and hence cannot mix with the special states [9]. This allows us to ‘integrate out’ all states other than the special states in (5.4) following the procedure described in [9], resum the series, and express (5.6) as

$$P\Pi(k)P = Z^{1/2}(k)(k^2 + M_p^2)^{-1}Z^{1/2}(k)^T, \quad (5.7)$$

where  $Z^{1/2}(k)$  is an  $n_p \times n_p$  matrix which has no poles near  $k^2 = -m^2$  and  $M_p$  is an  $n_p \times n_p$  diagonal matrix. The eigenvalues of  $M_p$  give physical renormalized masses of the special states and  $Z^{1/2}(k)$  evaluated at  $k^2 = -M_p^2$  correspond to wave-function renormalization factors.

### 5.3 Effect of changing the locations of picture changing operators

We shall now consider the change in the two point amplitude  $\mathcal{F}$  under an infinitesimal change in the locations of picture changing operators. For this we shall use (3.49). For now we shall proceed ignoring the boundary contribution described in the second term in (3.49), – its effect will be discussed separately in §5.5. The first term has two types of contributions – one where  $Q_B$  operates on the external vertex operator on the left and the other where  $Q_B$  operates on the external vertex operator on the right. Let us focus on the terms where  $Q_B$  acts on the external vertex operator on the left. Even though eventually we shall be interested in taking both external states to be special states, let us for now restrict only the external state on the left to be one of the  $n_p$  special states carrying momentum  $(k^0, \vec{k} = 0)$  described by a vertex operator of type  $W_+$  given in (5.1). Since  $Q_B W_+$  is proportional to  $(k^2 + m^2)$  we can take out a factor of  $(k^2 + m^2)$  – which is needed to cancel the external tree level propagator factor [9] – and call the rest of the contribution  $\delta\mathcal{H}$ . If  $n_p$  is the number of special states at a given mass level then  $\delta\mathcal{H}$  is  $n_p \times \infty$  dimensional matrix. Now by repeated use of (3.51) and (3.41)  $\delta\mathcal{H}$  can be decomposed into sum of products of 1PI contributions and propagators as in (5.4), but the

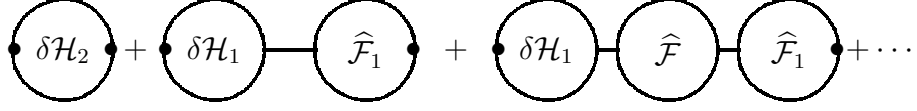


Figure 7: Pictorial representation of the expression for  $\delta\hat{\mathcal{H}}$  given in (5.8). The black dots denote external states without propagator factors.

effect of contraction with  $V_f$  in (3.49) will be to replace  $\Omega_{6g_i-6+2n_i}^{(g_i, n_i)}$  by  $\Omega_{6g_i-5+2n_i}^{(g_i, n_i)}[V_f]$  in *one of the* 1PI components, representing the effect of moving the picture changing operators on that particular component of the Riemann surface. In order to write down an expression like (5.4) for  $\delta\mathcal{H}$  we shall first define several new quantities:

1.  $\delta\mathcal{H}_1$  denotes the 1PI contribution to the two point function with one external state  $(k^2 + m^2)^{-1}Q_B W_+$  and the other external state arbitrary.
2.  $\delta\mathcal{H}_2$  will be defined in the same way as  $\delta\mathcal{H}_1$  but we replace  $\Omega_p^{(g, n)}$  by  $\Omega_{p+1}^{(g, n)}[V_f]$  in the integration measure.
3.  $\hat{\mathcal{F}}_1$  will be defined in the same way as  $\hat{\mathcal{F}}$ , but we replace  $\Omega_p^{(g, n)}$  by  $\Omega_{p+1}^{(g, n)}[V_f]$  in the integration measure (3.18).
4. Next we define

$$\delta\hat{\mathcal{H}} = \delta\mathcal{H}_2 + \delta\mathcal{H}_1\Delta\hat{\mathcal{F}}_1 + \delta\mathcal{H}_1\Delta\hat{\mathcal{F}}\Delta\hat{\mathcal{F}}_1 + \cdots = \delta\mathcal{H}_2 + \delta\mathcal{H}_1\Delta(1 - \hat{\mathcal{F}}\Delta)^{-1}\hat{\mathcal{F}}_1. \quad (5.8)$$

A pictorial representation of this has been shown in Fig. 7.

In terms of these quantities we can now express  $\delta\mathcal{H}$  as

$$\delta\mathcal{H} = \delta\hat{\mathcal{H}} + \delta\hat{\mathcal{H}}\Delta\hat{\mathcal{F}} + \delta\hat{\mathcal{H}}\Delta\hat{\mathcal{F}}\Delta\hat{\mathcal{F}} + \cdots = \delta\hat{\mathcal{H}}(1 - \Delta\hat{\mathcal{F}})^{-1} \quad (5.9)$$

which has been pictorially represented in Fig. 8. It is easy to see by inspection that Fig. 8, with  $\delta\hat{\mathcal{H}}$  defined via Fig. 7, sums over all possible contributions to  $\delta\mathcal{H}$ .

We shall now prove that  $\delta\hat{\mathcal{H}}$  has no poles at  $k^2 = -m^2$ . For this we examine each term on the right hand side of (5.8). Since  $\delta\mathcal{H}_2$  gets contribution from 1PI amplitudes, it has no poles. In the second term we could in principle get a pole at  $k^2 = -m^2$  from a state of momentum  $(k^0, \vec{k} = 0)$  propagating in  $\Delta$ . Let us denote the vertex operator of such a state by  $e^{ik_0 X^0} \mathcal{O}_g \tilde{V}$  where  $\mathcal{O}_g$  is some ghost sector operator and  $\tilde{V}$  is a matter sector operator with zero momentum.

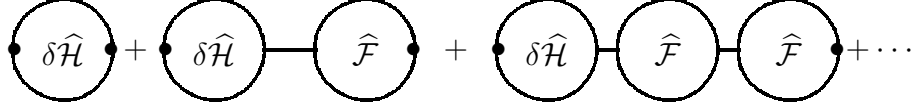


Figure 8: Pictorial representation for the expression for  $\delta\mathcal{H}$  given in (5.9).

In order to contribute to the pole at  $k^2 = -m^2$ ,  $\mathcal{O}_g\tilde{V}$  should have dimension  $(\alpha'm^2/4, \alpha'm^2/4)$  since  $e^{ik_0X^0}$  has dimension  $(\alpha'k^2/4, \alpha'k^2/4)$ . Now  $\delta\mathcal{H}_1$  has an insertion of  $Q_BW_+$  on the left for some special state vertex operator  $W_+$ . Since  $Q_BW_+$  is a vertex operator of ghost number 3 and picture number  $-1$ , by the conservation of ghost and picture number the states that contribute to  $\Delta$  must also have ghost number 3 and picture number  $-1$ . Since the states are annihilated by  $b_0$  and  $\bar{b}_0$ , the minimum dimension ghost sector operator with ghost number 3 and picture number  $-1$  is  $c\bar{c}\eta$  with total conformal weight  $-1$ . Thus in order for  $\mathcal{O}_g\tilde{V}$  to have dimension  $(\alpha'm^2/4, \alpha'm^2/4)$ ,  $\tilde{V}$  must have total dimension less than or equal to

$$1 + \alpha'\frac{m^2}{4} + \alpha'\frac{m^2}{4} = 1 + \alpha'\frac{m^2}{2}. \quad (5.10)$$

But by  $SO(D) \times G$  symmetry,  $\tilde{V}$  must belong to the same representation as  $W_+$  which is one of the  $R_i$ 's. Hence its conformal weight has a lower bound given in (5.2). We now see that (5.10) is inconsistent with (5.2). Hence there is no operator  $\tilde{V}$  that can contribute a pole to the second term on the right hand side of (5.8). Since  $\hat{\mathcal{F}}$  does not change the  $SO(D) \times G$  transformation law of the state, the same argument can be used to show that none of the factors of  $\Delta$  in the other terms on the right hand side of (5.8) can produce a pole at  $k^2 = -m^2$ . Thus  $\delta\hat{\mathcal{H}}$  is free from poles at  $k^2 = -m^2$ .

We now note that (5.4) and (5.9) are identical to eqs.(3.19) of [9], except that in [9]  $\delta$  was used to denote a change induced by a change in the choice of local coordinate system. Also the absence of poles in  $\delta\hat{\mathcal{H}}$  matches the similar property of  $\delta\hat{\mathcal{H}}$  in [9]. Thus from here on one can repeat arguments identical to those given in [9] to show that the change  $\delta\mathcal{H}$  in the two point function given in (5.9) can be absorbed into a change in the wave-function renormalization factor of the state associated with the special state vertex operator inserted on the left, and the renormalized masses do not change under a change of the locations of the picture changing operators. Similar analysis would establish that the effect of the terms in (3.49) in which  $Q_B$  acts on the external state on the right can be absorbed into a wave-function renormalization of the external state on the right.

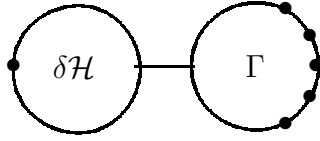


Figure 9: A contribution to the change in the S-matrix under the change in the location of the picture changing operators.

## 5.4 S-matrix

Similar argument can be used to show that the S-matrix of special states are also invariant under a change in the locations of the picture changing operators. For this we use (3.49) to manipulate the change in the  $n$ -point amplitude in a way similar to that for the two point amplitude. As in the case of mass renormalization, we postpone discussion of the boundary terms arising from the second term in (3.49) to §5.5. So we only have to examine the first term. This can be expressed as a sum of terms, in each of which  $Q_B$  acts on a particular external special state. If  $W$  denotes the vertex operator of this state then the term involving  $Q_B W$  can be manipulated in the same way as in the case of mass renormalization, expressing it as sums of products of amplitudes 1PI in momentum  $k$  and propagators carrying momentum  $k$ . The resulting contribution can be expressed as the sum of diagrams shown in Fig. 8 attached by a propagator  $\Delta$  to an amplitude  $\Gamma$  that is 1PI in momentum  $k$  and carries all other external states (see Fig. 9) plus a term without any pole near  $k^2 = -m^2$ . The term without pole does not contribute to the S-matrix. As in [9], the contribution of Fig. 9 can be shown to cancel against the change in the wave-function renormalization factor  $Z(k)^{-1/2}$  that appears in the expression for the S-matrix, establishing that the S-matrix is invariant under a change of the locations of the picture changing operators.

## 5.5 Boundary contributions

In this section we shall study the effect of possible boundary contributions corresponding to the second term on the right hand side of (3.49). The analysis of this section should be regarded as an iterative procedure since we need to use some of the results of §5.6 at lower genus / lower number of punctures. Also this analysis is independent of the nature and number of external states we have and holds for general external states.

As discussed above (3.52), possible sources of this boundary contribution are from special

degenerations where a given Riemann surface degenerates to two Riemann surfaces, with the momentum flowing through the degenerating punctures forced to vanish or satisfy tree level on-shell condition for some state. As long as we work in the off-shell formalism where the momenta of all the external states are generic and off-shell, the only possible contribution involves the case where the momentum flowing through the degenerating punctures vanishes. This requires that one of the component Riemann surfaces has no punctures (except the one corresponding to the degenerating puncture) and all the external states are inserted at the punctures of the other component.

The relevant boundary contribution is given in (3.52). Without any loss of generality we can take the surface  $\Sigma_1$  to be the one without any external puncture and the surface  $\Sigma_2$  to contain all the external punctures. Thus we have  $n_1 = 1$ . Now ghost and picture number conservation laws given in (3.20) tells us that in both terms on the right hand side of (3.52)  $|\varphi_i\rangle$  must carry picture number  $-1$  and ghost number  $3$  whereas  $|\varphi_j\rangle$  must have picture number  $-1$  and ghost number  $2$ . Furthermore the presence of  $b_0^-$  factors in the propagator tells us that both  $|\varphi_i^c\rangle$  and  $|\varphi_j^c\rangle$  must be annihilated by  $c_0^-$  and hence both  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  must be annihilated by  $b_0^-$ . Both  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  must be GSO even since the rest of the operators inserted in the correlations functions are GSO even. The integration over the angular variable  $\theta$  projects into states with  $L_0 = \bar{L}_0$ . Finally since eventually we shall take the upper cut-off  $\Lambda$  on the  $s$ -integral to infinity, the non-vanishing contributions to the boundary terms come from states with  $L_0 = \bar{L}_0 \leq 0$ . Thus we must classify all possible candidates for  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  consistent with these properties. In fact the choices of  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  are correlated due to the requirement that  $\langle\varphi_i^c|b_0^-|\varphi_j^c\rangle$  should be non-zero in order to get a non-vanishing contribution. This is best implemented by choosing a basis of states such that for each  $|\varphi_j\rangle$  satisfying the restrictions given above, there is a unique  $|\varphi_i\rangle$  for which  $\langle\varphi_j^c|b_0^-|\varphi_i^c\rangle$  is non-zero (or equivalently  $\langle\varphi_j|c_0^-|\varphi_i\rangle$  is non-zero). We shall now list the possible choices of  $|\varphi_i\rangle$  and  $|\varphi_j\rangle$  in heterotic string theory following [47]:

1.  $\varphi_j = c\bar{c}e^{-\phi}V$ ,  $\varphi_i = (\partial c + \bar{\partial}\bar{c})c\bar{c}e^{-\phi}V$  where  $V$  is a GSO even matter sector vertex operator of dimension  $(h + 1/2, h)$  with  $h \leq 1/2$ . Absence of tachyons in the spectrum of physical states tells us that the only operators of this kind present in the theory have dimension  $(1, 1/2)$ , and the corresponding  $\varphi_j$  describes a vertex operator of a zero momentum physical massless state. We shall assume that this represents a moduli field with vanishing potential – the case where the field has a potential can be analyzed

following the procedure described in [54]. In this case

$$\int_{\mathcal{M}_{g_1, n_1=1}} \Omega_{6g_1-6+2}^{(g_1, 1)}(|\varphi_j\rangle)|_{S_1} \quad (5.11)$$

represents a zero momentum tadpole of a physical massless state at genus  $g_1$ . Assuming that the vacuum we are working with is stable, this zero momentum tadpole must vanish and hence the second term on the right hand side of (3.52) vanishes after integration over  $\mathcal{M}_{g_1, 1}$ . On the other hand

$$\int_{\mathcal{M}_{g_2, n_2}} \Omega_{6g_2-6+2n_2}^{(g_2, n_2)}(|\Phi_2\rangle \otimes |\varphi_j\rangle)|_{S_2} \quad (5.12)$$

represents the effect of inserting a zero momentum massless external state in the genus  $g$  amplitude of the external states. This multiplied by any constant represents the effect of shifting the vacuum expectation value of the corresponding field by that constant. Thus for this choice of  $|\varphi_j\rangle$  the first term on the right hand side of (3.52) can be interpreted as the result of shifting the expectation value of this massless state by an amount proportional to [14, 47]

$$\int_{\mathcal{M}_{g_1, 1}} \Omega_{6g_1-5+2}^{(g_1, 1)}(|\varphi_i\rangle)[V_f^{(1)}]|_{S_1}. \quad (5.13)$$

This is turn can be interpreted as a field redefinition of the corresponding scalar field.

2.  $\varphi_j = \frac{1}{2}c\eta + \bar{c}\bar{\partial}^2\bar{c}c\partial\xi e^{-2\phi}$ ,  $\varphi_i = (\partial c + \bar{\partial}\bar{c})(\frac{1}{2}c\eta - \bar{c}\bar{\partial}^2\bar{c}c\partial\xi e^{-2\phi})$ . In this case we have

$$\varphi_j = \{Q_B, (\partial c + \bar{\partial}\bar{c})c\partial\xi e^{-2\phi}\}. \quad (5.14)$$

This is a pure gauge state. Thus by the result of §5.6 at a lower genus / lower number of punctures, (5.11) vanishes for this choice of  $\varphi_j$  whereas the effect of (5.12) can be absorbed into a wave-function renormalization of external states.

3.  $\varphi_j = \frac{1}{2}c\eta - \bar{c}\bar{\partial}^2\bar{c}c\partial\xi e^{-2\phi}$ ,  $\varphi_i = (\partial c + \bar{\partial}\bar{c})(\frac{1}{2}c\eta + \bar{c}\bar{\partial}^2\bar{c}c\partial\xi e^{-2\phi})$ . In this case  $|\varphi_j\rangle$  is a BRST invariant state and represents zero momentum dilaton in the  $-1$  picture. Thus (5.11) now has the interpretation of a dilaton tadpole. In a consistent background this must vanish. On the other hand (5.12) can be interpreted as the result of a zero momentum dilaton insertion into the amplitude. Thus for this choice of  $|\varphi_j\rangle$  the first term on the right hand side of (3.52) can be interpreted as the result of shifting the expectation value of the dilaton by an amount proportional to (5.13) [14, 47]. Equivalently we can regard this as a field redefinition of the dilaton field.

4.  $\varphi_j = (\partial c + \bar{\partial} \bar{c}) c \partial \xi e^{-2\phi} \bar{c} U$ ,  $\varphi_i = c \eta \bar{c} U$  where  $U$  is a dimension (1,0) GSO even operator in the matter sector. If the matter CFT has a discrete symmetry which changes the sign of  $U$  but leaves the super stress tensor invariant, then the terms in (3.52) involving correlation function on  $\Sigma_1$ , – the  $\Omega_{6g_1-5+2}^{(g_1,1)}(|\varphi_i\rangle)[V_f^{(1)}]|_{S_1}$  factor in the first term and the  $\Omega_{6g_1-6+2}^{(g_1,1)}(|\varphi_j\rangle)|_{S_1}$  in the second term – vanishes. This is so *e.g.* in ten dimensional flat space-time or toroidal compactification where  $U = \partial X^M$  for some compact or non-compact coordinate  $X^M$ , and there is always a symmetry that reverses the sign of  $X^M$  together possibly with some other  $X^N$ 's and their superpartners. Most known string compactifications have this property. Henceforth we shall restrict our analysis to those theories in which the contribution from this term to the right hand side of (3.52) vanishes.<sup>10</sup>
5.  $\varphi_j = (\partial c + \bar{\partial} \bar{c}) c e^{-\phi} V$ ,  $\varphi_i = c e^{-\phi} V \bar{c} \bar{\partial}^2 \bar{c}$  where  $V$  is a dimension (0, 1/2) GSO odd matter operator. Again if the matter CFT has a discrete symmetry under which  $V \rightarrow -V$  but the super-stress-tensor remains invariant then the terms in (3.52) involving correlation function on  $\Sigma_1$ , – the  $\Omega_{6g_1-5+2}^{(g_1,1)}(|\varphi_i\rangle)[V_f^{(1)}]|_{S_1}$  factor in the first term and the  $\Omega_{6g_1-6+2}^{(g_1,1)}(|\varphi_j\rangle)|_{S_1}$  in the second term – vanishes. This is so *e.g.* in ten dimensional flat space-time or toroidal compactification where  $V = \psi^M$  for some fermionic field  $\psi^M$  which is the superpartner of some compact or non-compact coordinate  $X^M$ , and there is always a symmetry that reverses the sign of  $(\psi^M, X^M)$  together possibly with some other  $X^N$ 's and their superpartners. Again most known string compactifications have this property. Henceforth we shall restrict our analysis to those theories for which this amplitude vanishes.

## 5.6 Decoupling of pure gauge states

An argument very similar to the one given in §5.3 can be used to show the vanishing of the S-matrix elements involving one or more pure gauge states of the form  $Q_B|\Lambda\rangle$  and the rest of the states corresponding to special states. Our starting point is again (3.30). Up to boundary terms which arise from the integration over total derivative term  $d\Omega_{p-1}^{(g,n)}$  and can be treated as in §5.5, we can transfer the BRST operator from over  $\Lambda$  to the other external states which we

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<sup>10</sup>At genus one the relevant correlator on the torus in the matter CFT involves just the one point function of the  $U(1)$  current  $U$ . Due to translational symmetry on the torus we can replace this by the contour integral of  $U$  along the  $a$ -cycle. In this case we can represent the correlator in the matter sector as a trace over all fields weighted by the  $U$ -charge and  $e^{2\pi i(\tau L_0 - \bar{\tau} \bar{L}_0)}$ . This receives equal and opposite contributions from the CPT conjugate states and hence always vanishes.

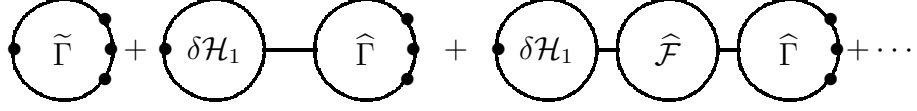


Figure 10: Result of organizing the amplitude with  $Q_B W$  as an external state into product of terms 1PI in momentum  $k$  and propagators carrying momentum  $k$ . The momentum  $k$  enters through the external state represented by the left black dot in each diagram.

have assumed to be special states. Let us denote the vertex operator of such a special state by  $W$  and the momentum carried by it by  $k$  and analyze the term involving the insertion of  $Q_B W$  to the amplitude. We can decompose the amplitude into sum of products of terms which are 1PI in the momentum  $k$  and propagators carrying momentum  $k$ . The resulting organization of the amplitude takes the form shown in Fig. 10, with  $\hat{\Gamma}$  denoting a component that is 1PI in momentum  $k$  and carries all other external states and  $\tilde{\Gamma}$  denoting a contribution to the full amplitude that is 1PI in momentum  $k$  and carries all external states.  $\delta\mathcal{H}_1$  is the same quantity that appears in §5.3. The same argument as before now tells us that none of these diagrams have any poles at  $k^2 + m^2 = 0$  where  $m$  is the tree level mass of the special states, and hence after adding them we do not get a pole at  $k^2 = -m_p^2$ . Thus the contribution from these diagrams to the S-matrix vanishes. The boundary terms can be analyzed as in §5.5 with  $\Lambda$  insertion now playing the role of contraction with  $V_f$ . There is an added simplification in that as long as  $\Lambda$  carries generic momentum, it can only be inserted on the surface  $\Sigma_2$  where the rest of the vertex operators are inserted. Thus the analog of the terms in (3.52) involving  $V_f^{(1)}$  will be absent.

The same arguments can be used to show that the insertion of (5.14) to an amplitude gives vanishing contribution to the S-matrix element, but a few additional terms need to be analyzed. First of all, since (5.14) carries zero momentum, the term containing  $Q_B W$  insertion to the amplitude can have poles from diagrams shown in the second line of Fig. 11. However as discussed in the caption of this figure, the effect of these terms can be absorbed into a redefinition of the wave-function renormalization factor of the special state described by the vertex operator  $W$ . The second difference is that in the boundary terms, the operator  $\Lambda = -(\partial c + \bar{\partial} \bar{c})c\partial\xi e^{-2\phi}$  that is left after stripping off  $Q_B$  from (5.14) can be inserted into  $\Sigma_1$  as well as on  $\Sigma_2$  and hence we have the analog of both terms that appear in (3.52). This however does not pose any difficulty since the boundary terms may be analyzed in the same way as in §5.5.



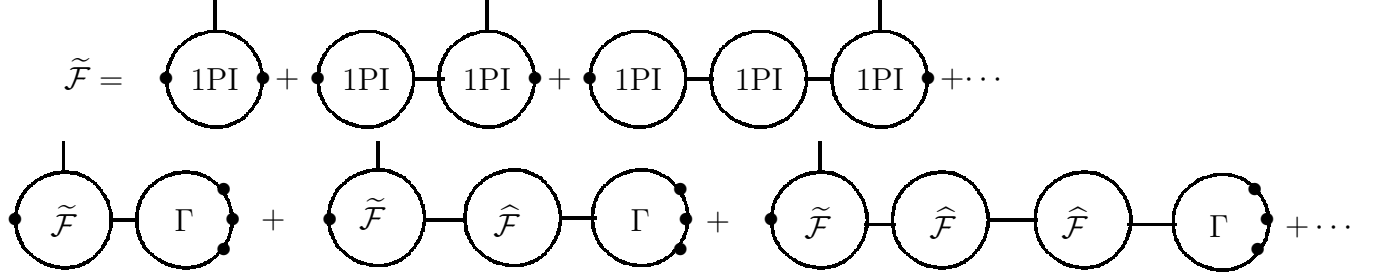


Figure 11: The first line gives the definition of  $\tilde{\mathcal{F}}$  with the external left black dot in each graph denoting the insertion of  $Q_B W$  carrying momentum  $k$  and the vertical external line on top denoting the insertion of the zero momentum vertex operator  $\Lambda = (\partial c + \bar{\partial} \bar{c}) c \partial \xi e^{-2\phi}$  that appears in (5.14). It follows from the analysis similar to that for  $\delta \hat{\mathcal{H}}$  in §5.3 that  $\tilde{\mathcal{F}}$  has no poles near  $k^2 + m^2 = 0$  despite the appearance of internal propagators carrying momentum  $k$  from the second term onwards. The second line gives the diagrams that could generate poles near  $k^2 + m^2 = 0$  in the amplitude containing  $Q_B W$ ,  $\Lambda$  and other special states as external legs. Here  $\Gamma$  is 1PI in momentum  $k$  and carries all external states other than  $Q_B W$  and  $\Lambda$ . Since the net contribution of the second line can be interpreted as the result of multiplying the external propagator of the Green's function by  $\tilde{\mathcal{F}}$  from the left, the effect of this is to change the wave-function renormalization factor  $Z^{1/2}$  of the special state described by the vertex operator  $W$  by a factor proportional to  $\tilde{\mathcal{F}}$ .

## 5.7 Type II string theory

The analysis for type II string theory proceeds in the same way except that the list of operators  $\varphi_i$  and  $\varphi_j$  which can appear in the sum over states in (3.52) are different. We shall now analyze their contributions.

1.  $\varphi_j = c \bar{c} e^{-\phi} e^{-\bar{\phi}} V$ ,  $\varphi_i = (\partial c + \bar{\partial} \bar{c}) c \bar{c} e^{-\phi} e^{-\bar{\phi}} V$  where  $V$  is a left-GSO and right-GSO odd matter sector vertex operator of dimension  $(h, h)$  with  $h \leq 1/2$ . Absence of tachyons in the theory tells us that the only possible value of  $h$  is  $1/2$  in which case  $\varphi_j$  describes the vertex operator of a physical massless state. We can then follow the analysis used in the case of heterotic string theory to show that as long as tadpoles of massless fields vanish, boundary contributions involving these operators can be absorbed into a shift of the vacuum expectation values of the massless fields.
2.  $\varphi_j = \frac{1}{2}(c \bar{\eta} \bar{c} \bar{\partial} \xi e^{-2\bar{\phi}} + \bar{c} \eta c \partial \xi e^{-2\phi})$ ,  $\varphi_i = \frac{1}{2}(\partial c + \bar{\partial} \bar{c})(c \bar{\eta} \bar{c} \bar{\partial} \xi e^{-2\bar{\phi}} + \bar{c} \eta c \partial \xi e^{-2\phi})$ . In this case we have

$$\varphi_j = \{Q_B, (\partial c + \bar{\partial} \bar{c}) c \partial \xi e^{-2\phi} \bar{c} \bar{\partial} \xi e^{-2\bar{\phi}}\}. \quad (5.15)$$

This is a pure gauge state. Thus as in the case of heterotic string theory, its effect can be absorbed into a wave-function renormalization of external states.

3.  $\varphi_j = \frac{1}{2}(c\eta\bar{c}\bar{\partial}\bar{\xi}e^{-2\bar{\phi}} - \bar{c}\bar{\eta}c\partial\xi e^{-2\phi})$ ,  $\varphi_i = \frac{1}{2}(\partial c + \bar{\partial}\bar{c})(c\eta\bar{c}\bar{\partial}\bar{\xi}e^{-2\bar{\phi}} - \bar{c}\bar{\eta}c\partial\xi e^{-2\phi})$ . In this case  $|\varphi_j\rangle$  is a BRST invariant state and represents zero momentum dilaton in the  $(-1, -1)$  picture. Thus as in the case of heterotic string theory, the effect of this term can be absorbed into a shift in the expectation value of the zero momentum dilaton field.
4.  $\varphi_j = (\partial c + \bar{\partial}\bar{c})c\partial\xi e^{-2\bar{\phi}}\bar{c}e^{-\bar{\phi}}U$ ,  $\varphi_i = c\eta\bar{c}e^{-\bar{\phi}}U$  where  $U$  is a left GSO odd, right GSO even dimension  $(1/2, 0)$  operator in the matter sector. In a unitary theory such an operator is a superconformal primary. As in the case of heterotic string theory, if the theory has a discrete symmetry under which  $U \rightarrow -U$  keeping the super-stress tensor invariant, then the matrix element of this operator on  $\Sigma_1$  will vanish. We shall restrict our analysis to the class of theories for which this holds..
5.  $\varphi_j = (\partial c + \bar{\partial}\bar{c})\bar{c}\bar{\partial}\bar{\xi}e^{-2\bar{\phi}}ce^{-\phi}V$ ,  $\varphi_i = \bar{c}\bar{\eta}ce^{-\phi}V$  where  $V$  is a left GSO even, right GSO odd dimension  $(0, 1/2)$  operator in the matter sector. This case can be treated exactly as the previous one with the roles of left and right moving sectors on the world-sheet exchanged.

## 5.8 General states

For external states which are not special, we need to choose a suitable basis for physical, unphysical and pure gauge states, ‘diagonalize’ the propagator at each mass level after ‘integrating out’ fields at other mass levels and identify the renormalized physical states and their masses. For bosonic string theory this procedure has been described in detail in [10]. We expect that there should not be any surprises in generalizing the analysis of [10] to heterotic or type II string theory, but we shall postpone a detailed analysis of this question to the future.

## 6 Ramond sector

Let us now consider the case where the external vertex operators also include (an even number of) Ramond punctures. If we take all the external Ramond punctures in the  $-1/2$  picture then on a genus  $g$  surface with  $n_B$  NS punctures and  $2n_F$  Ramond punctures, we need a total of  $2g - 2 + n_B + n_F$  picture changing operators. With this change we can define the off-shell amplitudes in the same way as in §3.

## 6.1 The problem

The problem however appears while choosing a gluing compatible assignment of picture changing operators near a degeneration where a Ramond sector state propagates along the tube connecting the Riemann surfaces. Consider for example the case where the Riemann surface described above degenerates into a pair of Riemann surfaces of genus  $g_1$  and  $g_2$  with the first one carrying  $n_{B_1}$  NS punctures and  $2n_{F_1}$  R-punctures and the second one carrying  $n_{B_2}$  NS-punctures and  $2n_{F_2}$  R-punctures, satisfying

$$g = g_1 + g_2, \quad n_B = n_{B_1} + n_{B_2}, \quad 2n_F = 2n_{F_1} + 2n_{F_2} - 2. \quad (6.1)$$

From the fact that the total number of external Ramond states decreases by 2 after gluing we know that the vertex operator at the punctures being glued must be Ramond vertex operators. However in this case since the picture numbers of the Ramond vertex operators at these punctures must add up to  $-2$ , it is not possible to take both of them in the  $-1/2$  picture as in the case of external states. Indeed we can see that we run into an apparent contradiction if we take both in the  $-1/2$  picture. In that case on the first Riemann surface we shall have  $2g_1 - 2 + n_{B_1} + n_{F_1}$  picture changing operators and on the second Riemann surface we shall have  $2g_2 - 2 + n_{B_2} + n_{F_2}$  picture changing operators. Using (6.1) their numbers add up to  $2g + n_B + n_F - 3$ . This is one less than the number of picture changing operators which were inserted on the original surface.

This shows that it is impossible to satisfy the gluing compatibility condition in its original form which required that the arrangement of picture changing operators on the glued surface must agree with the collection of picture changing operators on the individual surfaces.

## 6.2 The prescription

We shall now suggest a possible procedure which is not elegant but practical. This procedure essentially specifies the rules for building up the full amplitude from 1PI amplitudes.

1. For computing 2-point amplitude of two Ramond states, we take one of them in the  $-1/2$  picture and the other one in the  $-3/2$  picture. In this case near any Ramond degeneration of this amplitude, we pick the vertex operator at the degenerating puncture that is closer to the  $-1/2$  picture vertex operator in the  $-3/2$  picture and the other one in the  $-1/2$  picture. This basically means that if the degeneration is into a genus  $g_1$  and

a genus  $g_2$  surface then we take  $2g_1$  picture changing operators to lie on the genus  $g_1$  surface and  $2g_2$  picture changing operators to lie on the genus  $g_2$  surface.

2. For  $n$ -point amplitude with  $n \geq 3$  we take all external Ramond sector states in the  $-1/2$  picture and adopt the following algorithm. For any given set of  $n$  external states, we declare all subsets of length  $\leq (n-1)/2$  as **B**-type, all subsets of length  $\geq (n+1)/2$  as **A**-type and (if  $n$  is even) classify the subsets of length  $n/2$  arbitrarily as **A**-type or **B**-type, with the constraint that the complement of an **A**-type set is always **B**-type and vice versa. Now for any given Ramond degeneration of the original punctured Riemann surface into a pair of punctured Riemann surfaces, one side will contain an **A**-type set of external states and the other side will contain a complementary set that is of **B**-type. Our prescription will be that on the component that contains external states in the **A**-type set we choose the state at the degenerating puncture to be a picture number  $-1/2$  state, while on the component that contains external states in the **B**-type set we choose the state at the degenerating puncture to be a picture number  $-3/2$  state. This effectively means that the extra picture changing operator is inserted on the component of the Riemann surface that contains external states in the **B**-type set.

Clearly the division of the external states into **A** and **B**-type is arbitrary. The important point is that the rules for making such divisions, although arbitrary, must *depend only on how the external states are divided up into subsets by the degeneration and not on the genera or the moduli of the Riemann surfaces involved in the degeneration*. This is necessary for the factorization property of the amplitude to be discussed in §6.3.

This procedure implies that the 1PI amplitudes which are glued together to form the full amplitude can some time have all the external states in the  $-1/2$  picture, and other times one of their external states in the  $-3/2$  picture. The fact that any subset of a **B**-type set is always **B**-type guarantees that we do not get any 1PI amplitude with more than one  $-3/2$  picture vertex operator. Gluing compatibility then tells us that for any such 1PI amplitude, the arrangement of picture changing operators on the corresponding Riemann surface (and of course the choice of local coordinates at the punctures) must be chosen in a way that only depends on the moduli relevant to that 1PI amplitude and is independent of the rest of the Riemann surfaces to which it is attached by plumbing fixture.

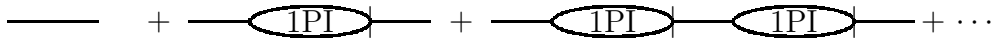


Figure 12: Pictorial representation of eq.(6.3). The horizontal lines represent  $\Delta$  and blobs marked 1PI represent  $\tilde{\Gamma}$ . Note that this diagram is left-right asymmetric as in each 1PI component the state on the left is in the  $-1/2$  picture and the state on the right is in the  $-3/2$  picture. The latter has been indicated by a  $|$  at the vertex.

### 6.3 Analysis of propagator and S-matrix

We shall now briefly describe how this prescription helps us define propagators and S-matrix elements. Let us start with the Ramond sector propagator where one of the external vertex operators is in the  $-1/2$  picture and the other one is in the  $-3/2$  picture. Let  $k$  be the momentum carried by the  $-1/2$  picture external vertex operator. In this case the tree level propagator is

$$\Delta = (L_0 + \bar{L}_0)^{-1} \delta_{L_0, \bar{L}_0}. \quad (6.2)$$

We now follow the procedure of [9, 10] to divide the higher genus amplitudes into one particle reducible (1PR) and one particle irreducible (1PI) amplitudes. Each 1PI component will have one external vertex operator carrying momentum  $k$  in the  $-1/2$  picture and the other external vertex operator carrying momentum  $-k$  in the  $-3/2$  picture. If  $\tilde{\Gamma}$  denotes the 1PI amplitude then the full propagator can be written as

$$\Pi = \Delta + \Delta \tilde{\Gamma} \Delta + \Delta \tilde{\Gamma} \Delta \tilde{\Gamma} \Delta + \dots = (\Delta^{-1} - \tilde{\Gamma})^{-1}. \quad (6.3)$$

This has been shown pictorially in Fig. 12. The poles of this give the renormalized mass<sup>2</sup>. Following the procedure described in [9, 10] one can further simplify this by integrating out states at all mass levels except a particular level and reduce the problem to that of diagonalization of a finite dimensional matrix, but we shall not discuss the details here.

Note that the familiar  $G_0$  in the numerator is missing from the tree level propagator  $\Delta$  that appears in (6.3). This is related to choosing specific normalization of the basis states i.e. the choice  $\langle -3/2, r | c_0 \bar{c}_0 | -1/2, s \rangle = \delta_{rs}$  that has been used to get the propagator  $\Delta$ . As a result the  $G_0$  factor is hidden inside  $\tilde{\Gamma}$  in (6.3). At the end of this section we shall describe how this factor can be recovered explicitly when the propagating state is a special state in the spirit described in §5.1.

Now consider the case of a general amplitude with external fermions with momenta  $k_1, \dots, k_n$

and *tree level* masses  $m_1, \dots, m_n$ . One would like such an amplitude to satisfy the following properties:

1. After multiplying the amplitude by a factor of  $\Delta$  for each external leg to generate the off-shell Green's function as described in (1.1), each amplitude should have an explicit factor of the full propagator  $\Pi(k_i)$  for each external leg. This will allow us to compute the S-matrix element using the LSZ prescription.
2. If  $k$  denotes the sum of a subset of the external momenta then the pole of the S-matrix in the  $k^2$  plane must come from the poles of  $\Pi(k)$ .

Let us begin with the first property. We break up the amplitude into sums of products of 1PI amplitudes as usual. Since subsets containing single external fermions are always **B**-type, the structure of self energy corrections associated with each of the external Ramond sector states will be the same as the one described in Fig. 12 for the two point function. In particular in any 1PI subamplitude inserted on the  $i$ -th external leg, we always pick the  $-3/2$  picture vertex operator on the puncture carrying momentum  $-k_i$  and  $-1/2$  picture vertex operator on the puncture carrying momentum  $k_i$ . Thus we get precisely the same factor (6.3) for each external leg. From this we can compute the on-shell S-matrix elements using the LSZ prescription as in [9, 10].

A similar analysis can be used to prove the second property. To find the pole of the amplitude in  $k^2$  we can express the off-shell amplitude  $\Gamma$  as sums of 1PI and 1PR contributions in legs carrying momentum  $k$  as

$$\Gamma = \widehat{\Gamma} + \widehat{\Gamma}_1^a \Pi_{ab} \widehat{\Gamma}_2^b, \quad (6.4)$$

where  $\widehat{\Gamma}$  represents contributions to  $\Gamma$  which are 1PI in the leg carrying momentum  $k$  and  $\widehat{\Gamma}_1^a$ ,  $\widehat{\Gamma}_2^b$  are also subamplitudes 1PI in momentum  $k$ . A pictorial representation of the second term on the right hand side of (6.4) has been shown in Fig. 13. Between  $\widehat{\Gamma}_1^a$  and  $\widehat{\Gamma}_2^b$ , one of them carries external states in **A**-type set and the other carries external states in **B**-type set. In the former the internal state will be inserted using  $-1/2$  picture while in the latter it will be inserted in the  $-3/2$  picture. This makes it manifest that the poles in the amplitude as a function of  $k^2$  occur exactly at the poles of  $\Pi$  given in (6.3).

Following the analysis of [9, 10] and the ones carried out here, one may be able to prove that the renormalized masses and S-matrix elements computed this way are independent of the detailed arrangement of picture changing operators as well as of how we assign subsets of

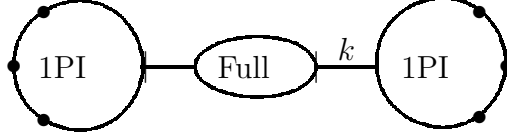


Figure 13: Pictorial representation of the second terms on the right hand sides of eq.(6.4). Here 1PI means sum of contributions which are 1PI in the leg carrying momentum  $k$ , whereas Full means sum of all contributions to the 2-point function shown in Fig. 12. If external states on the left side belong to the **B**-type set, then the internal state on the left side inserted into the 1PI amplitude is in the  $-3/2$  picture while the internal state on the right inserted into the 1PI amplitude is in the  $-1/2$  picture. As in Fig. 12, the  $-3/2$  picture state insertions are marked by  $|$ .

$1, \dots, n$  to be of **A**-type and **B**-type. A detailed analysis of this will be postponed to future work. However at this stage we note that for special Ramond sector states – in the spirit described in §5.1 – one can recover the symmetry between the **A**-type and **B**-type sets as follows. We simply require that in any degeneration limit the extra picture changing operator that is inserted on the part of the amplitude with external states in the **B**-type set approaches the degeneration node that carries the  $-3/2$  picture vertex operator.<sup>11</sup> Since for special states all the relevant vertex operators have the form  $\bar{c} c e^{-3\phi/2} \tilde{V}_\alpha e^{ik \cdot X}(z)$  for some matter sector vertex operator  $\tilde{V}_\alpha$ , the only term in the picture changing operator that gives a non-zero contribution in this limit is the  $e^\phi T_F(w)$  term in  $\mathcal{X}(w)$ . Now for special states, there are no matter sector operators carrying the same weight as  $\tilde{V}_\alpha$  and having conformal weight less than that of  $\tilde{V}_\alpha$ . Thus the maximum singularity we can get from the operator product of  $T_F(w)$  and  $\tilde{V}_\alpha(z)$  is  $(w-z)^{-3/2}$ ,  $3/2$  being the conformal weight of  $T_F$ . This cancels with the  $(w-z)^{3/2}$  factor from the operator product of  $e^\phi$  and  $e^{-3\phi/2}$ , producing a non-singular term.<sup>12</sup> Its contribution is

$$\lim_{w \rightarrow z} e^\phi T_F(w) \bar{c} c e^{-3\phi/2} \tilde{V}_\alpha e^{ik \cdot X}(z) \propto \bar{c} c e^{-\phi/2} (\gamma^\mu k_\mu \pm M)_\alpha{}^\beta V_\beta(z) \quad (6.5)$$

<sup>11</sup>This is equivalent to inserting the picture changing operator on the propagator.

<sup>12</sup>The condition for being able to do this is actually less stringent than a special state condition. The latter requires restriction on the conformal weight in both the holomorphic and the antiholomorphic sectors, while here we only need restriction on the holomorphic conformal weight.

where  $M$  is the tree level mass of the vertex operator and  $V_\beta$  is another matter sector operator that appears in the expression for the  $-1/2$  picture vertex operator. The  $(\gamma^\mu k_\mu \pm M)_\alpha{}^\beta$  factor comes from the action on  $\tilde{V}_\beta$  of the  $G_0$  term in the mode expansion of  $T_F$ . This converts the  $-3/2$  picture vertex operator to  $-1/2$  picture vertex operator and converts the propagator  $\delta_\alpha{}^\beta/(k^2 + M^2)$  to  $(\gamma^\mu k_\mu \pm M)_\alpha{}^\beta/(k^2 + M^2)$  which is the correct Ramond sector propagator. Since now all Ramond degenerating nodes carry  $-1/2$  picture vertex operators, we recover the symmetry between the external states in the **A** and **B**-type set.

## 7 Computation of Fayet-Iliopoulos terms using picture changing operator

The procedure for defining off-shell amplitudes using picture changing operators, as described above, can also be used for on-shell amplitudes. In this section we shall describe how it can be used to compute the effect of Fayet-Iliopoulos (FI) terms in SO(32) heterotic string theory compactified on a Calabi-Yau 3-fold [34–41].

### 7.1 Choice of locations of picture changing operators and local coordinate system

We refer the reader to the original papers for the necessary background, and focus here only on the computational aspect of the problem. The problem involves computing an on-shell two point function of two NS sector states at one loop order. The vertex operators describing the states have the form

$$\bar{c} c e^{-\phi} V_1 e^{ik \cdot X} \quad \text{and} \quad \bar{c} c e^{-\phi} V_2 e^{-ik \cdot X}, \quad k^2 = 0, \quad (7.1)$$

where  $X^\mu$  for  $0 \leq \mu \leq 3$  denote the four non-compact target space-time coordinates and  $V_1, V_2$  are a pair of superconformal primaries of dimension  $(1, 1/2)$  made of the degrees of freedom associated with the compact directions. Some of the special properties of  $V_i$  that we shall need will be reviewed later as and when we need them. In this section we shall work in the  $\alpha' = 1$  unit in which  $X^\mu$  and its fermionic partner  $\psi^\mu$  have the following operator product:

$$\partial X^\mu(z) \partial X^\nu(w) = -\frac{\eta^{\mu\nu}}{2(z-w)^2} + \cdots, \quad \psi^\mu(z) \psi^\nu(w) = -\frac{\eta^{\mu\nu}}{2(z-w)} + \cdots, \quad (7.2)$$



where  $\dots$  denote non-singular terms. The matter energy momentum tensor  $T(z)$  and its superpartner  $T_F(z)$  have the following form

$$T(z) = -\partial X^\mu \partial X^\nu \eta_{\mu\nu} + \psi_\mu \partial \psi^\mu + T_{int}, \quad T_F(z) = -\psi_\mu \partial X^\mu + (T_F)_{int}, \quad (7.3)$$

where the subscript  $_{int}$  denotes contributions from the compact directions.

For computing this amplitude we need to first compute the correlation function on the two punctured torus with the two vertex operators given in (7.1) inserted at the punctures, together with appropriate insertion of ghost fields and the picture changing operators as described in §3, and then integrate this over the moduli space of two punctured tori. The latter will be parametrized by the coordinates of one of the punctures (with the other one kept fixed) and the modular parameter  $\tau$  of the torus. Let  $u$  be the coordinate system in which the torus is described by the identification

$$u \equiv u + m + n\tau, \quad m, n \in \mathbb{Z}. \quad (7.4)$$

Then we shall choose the punctures  $P_1$  and  $P_2$  to be at  $u = 0$  and  $u = y$  and use  $y$  and  $\tau$  as complex coordinates of the moduli space. Furthermore we shall use

$$w_1 = u, \quad \text{and} \quad w_2 = u - y, \quad (7.5)$$

as the local coordinates around the puncture  $P_1$  and  $P_2$  (up to overall phases).<sup>13</sup>

Let us now describe the choice of locations of the picture changing operators. We need two of them. We shall choose one to be at a fixed location in the  $u$  coordinate, say at  $u = u_1$  and the other one at a location  $y = u_2 \equiv \alpha y$  for some fixed constant  $\alpha$ . This has the property that in the degeneration limit when  $y \rightarrow 0$ , if we regard the configuration as a three punctured sphere glued to a one punctured torus (with  $u/y$  as the coordinate system on the sphere and  $u$  as the coordinate system on the torus), the picture changing operator at  $u_1$  lies on the torus

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<sup>13</sup>This choice is not quite gluing compatible, as the latter requires that as we take the degeneration limit  $y \rightarrow 0$ , the distance between the punctures should remain fixed in the local coordinate system of each puncture. This can be achieved by scaling the local coordinates by  $1/y$ . Following the analysis of §2 one can show that this will have two effects. First of all it will generate an extra factor of  $|y|^{h_1+h_2}$  where  $h_1$  and  $h_2$  are the total conformal weights of the vertex operators inserted at the punctures. Since for on-shell external vertex operators we have  $h_1 = 0$ ,  $h_2 = 0$ , this effect disappears. Second, since the relationship between the local coordinates and the fixed coordinate  $u$  vary by a scale factor as we vary  $y$ , we shall have additional insertions in the correlation function involving  $b_0$  and  $\bar{b}_0$  operators acting on external states. However since the external states satisfy the Siegel gauge condition  $b_0|\Psi\rangle = \bar{b}_0|\Psi\rangle = 0$ , this effect also disappears. Thus we can continue to use  $u$  and  $u - y$  as the local coordinates.

and the picture changing operator at  $u_2$  lies on the sphere. This is the correct prescription for gluing compatibility. We could also have made a non-holomorphic choice in which  $u_2$  is taken to be a function of  $y$  and  $\bar{y}$ . We have not done this in order to keep the analysis simple, but the final result remains the same even for this more general choice.

Once we have made a choice of local coordinates and the location of the picture changing operators, we have fixed the choice of the section in  $\tilde{\mathcal{P}}_{1,2}$  on which we shall integrate. This means that the tangent vectors  $\partial/\partial\tau$ ,  $\partial/\partial y$  and their barred counterparts now have definite images in the tangent space of  $\tilde{\mathcal{P}}_{1,2}$ , and the integration measure will be given by the contraction of  $\Omega_4^{1,2}$  with these tangent vectors. This in turn means that we should now be able to use (3.19) to write down explicitly the measure that needs to be integrated over  $\mathcal{M}_{1,2}$  to compute the relevant amplitude. We shall now do this explicitly. For this we shall follow the general procedure described in §2.6 and §3.4 in which we divide the Riemann surface into different components with different coordinate systems and the functional relationship between the coordinates encode information about the moduli. We choose three different coordinate systems on the torus: the coordinate  $u$  introduced earlier which is also the local coordinate  $w_2$  around the puncture at  $y = 0$ , the local coordinate  $w_1 = u - y$  defined around the puncture at  $y$ , and a new coordinate system  $z$  defined as follows. Around the line  $C_2$ :  $\text{Im } u = a$  for some positive constant  $a$  we have  $z = u - \tau$  and along the line  $C_3$ :  $\text{Im } u = -b$  for some negative constant  $-b$  we have  $z = u$ . Then the whole torus, whose fundamental domain we shall take to be the region

$$-b \leq \text{Im } u < \tau_2 - b, \quad -\frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } u \leq \text{Re } u < \frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } u, \quad (7.6)$$

is covered by three regions. Around the puncture at  $y$  we identify a small disk  $D_1$  inside which we use the  $w_1 = u - y$  coordinate system. In the region

$$D_2 : -b \leq \text{Im } u < a, \quad -\frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } u \leq \text{Re } u < \frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } u, \quad u \notin D_1, \quad (7.7)$$

we use the  $u$  coordinate system. Finally in the region

$$D_3 : a - \tau_2 \leq \text{Im } z < -b, \quad -\frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } z \leq \text{Re } z < \frac{1}{2} + \frac{\tau_1}{\tau_2} \text{Im } z, \quad (7.8)$$

we use the  $z$  coordinate system. This has been shown in Fig. 14. The functional relationship between the coordinates takes the form

$$\begin{aligned} \text{On } C_1 = \partial D_1 : \quad w_1 &= u - y, \\ \text{On } C_2 : z &= u - \tau, \\ \text{On } C_3 : z &= u. \end{aligned} \quad (7.9)$$

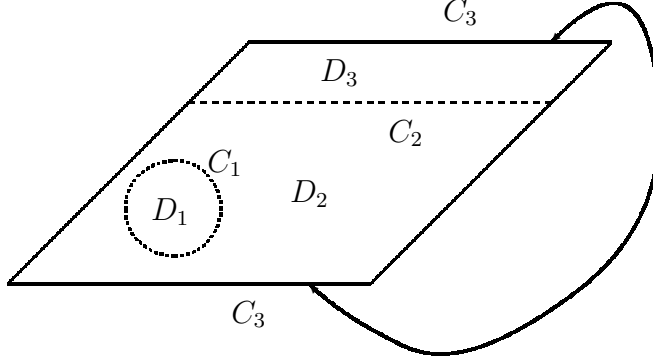


Figure 14: The covering of the torus by three different regions.

Both picture changing operators will be introduced in the  $u$  coordinate system in the region  $D_2$ .

## 7.2 The integration measure

Let us now follow the prescription of §2.6 and §3.4 to determine the insertion of  $b$ -ghost and picture changing operators. First let us ignore the picture changing operators and determine the  $b$ -ghost insertions as if we were working in bosonic string theory. In this case using (7.9) and the analysis of §2.6 we see that the effect of contraction of  $\Omega_p^{1,2}$  with  $\partial/\partial\tau$  is a  $b$ -ghost insertion along  $C_2$  since the only dependence on  $\tau$  arises from the gluing function along  $C_2$ . Furthermore since for fixed  $y$ ,  $z_{\tau+\delta\tau} = z_\tau - \delta\tau$ , the associated vector field is  $v(z) = -1$ , showing that the  $b$ -ghost insertion corresponding to contraction with  $\partial/\partial\tau$  and  $\partial/\partial\bar{\tau}$  is just

$$(-b_\tau)(-\bar{b}_\tau), \quad b_\tau \equiv \oint_{C_2} dz b(z), \quad \bar{b}_\tau \equiv \oint_{C_2} d\bar{z} \bar{b}(\bar{z}). \quad (7.10)$$

In the definition of  $b_\tau$  ( $\bar{b}_\tau$ ) the integration contour  $C_2$  should be oriented so that the region  $D_3$  lies to its left (right). On the other hand since the only  $y$  dependence of the gluing function is along the curve  $C_1$ , the effect of contraction with  $\partial/\partial y$  is represented by a contour integral along  $C_1$ . The associated vector field is  $v(w_1) = -1$  and hence we have the insertion of

$$(-b_y)(-\bar{b}_y), \quad b_y \equiv \int_{C_1} dw_1 b(w_1), \quad \bar{b}_y \equiv \int_{C_1} d\bar{w}_1 \bar{b}(\bar{w}_1). \quad (7.11)$$

In the definition of  $b_y$  ( $\bar{b}_y$ ) the integration contour runs anticlockwise (clockwise) around the puncture at  $y$ . The  $-$  signs in (7.10) and (7.11) reflect the  $-$  signs in the vector fields associated with  $\tau$  and  $z$  deformation.

Now let us consider the effect of inserting the picture changing operators. Both of them are inserted in the region  $D_2$  in the  $u$  coordinate system. Of them one location  $u_1$  is fixed while the other one  $u_2$  varies with  $y$  as  $\alpha y$ . It follows from the general prescription of §3.4 that the net insertion of  $b$ -ghosts and picture changing operators into the correlation function will be

$$b_\tau \bar{b}_\tau \mathcal{X}(u_1) (\mathcal{X}(u_2)(-b_y) - \partial \xi(u_2)(\partial u_2 / \partial y))(-\bar{b}_y) \quad (7.12)$$

Using  $u_2 = \alpha y$  and the form of the vertex operators given in (7.1) we can now write the net contribution to the torus 2-point function as<sup>14</sup>

$$\int d^2\tau \int d^2y \left\langle b_\tau \bar{b}_\tau \mathcal{X}(u_1) \bar{c}(0) c(0) e^{-\phi(0)} V_1(0) e^{ik \cdot X(0)} \right. \\ \left. \left( \mathcal{X}(\alpha y) + \alpha \partial \xi(\alpha y) c(y) \right) e^{-\phi(y)} V_2(y) e^{-ik \cdot X(y)} \right\rangle, \quad (7.13)$$

where  $\partial$  always denotes derivative with respect to the argument. The final result should be independent of  $\alpha$ . This will be verified explicitly in §7.4.

### 7.3 Evaluation of the amplitude

We shall now evaluate (7.13). First we write

$$\mathcal{X}(u_1) = \mathcal{X}(0) + (\mathcal{X}(u_1) - \mathcal{X}(0)) = \mathcal{X}(0) + \{Q_B, \xi(u_1) - \xi(0)\}, \quad (7.14)$$

where we have used (3.7). The contribution from the first term can be analyzed by noting that

$$\mathcal{X}(0) \bar{c}(0) c(0) e^{-\phi(0)} V_1(0) e^{ik \cdot X(0)} = \bar{c}(0) c(0) \left\{ \tilde{V}_1(0) - \frac{i}{2} k \cdot \psi(0) V_1(0) \right\} e^{ik \cdot X(0)} \\ - \frac{1}{4} \bar{c}(0) \eta(0) e^{\phi(0)} V_1(0) e^{ik \cdot X(0)}, \quad (7.15)$$

where  $\psi^\mu$  denote the world-sheet fermions describing superpartners of the non-compact target space-time coordinates  $X^\mu$ , and  $\tilde{V}_1$  is the dimension (1,1) vertex operator of the conformal

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<sup>14</sup>In this expression we have dropped an overall normalization of  $1/\pi^2$  that comes from a combination of two terms. First the  $(2\pi i)^{-(3g-3+n)}$  factor in (3.18) gives a factor of  $-1/4\pi^2$ . Second the integration measure is really  $d\tau \wedge d\bar{\tau} \wedge dy \wedge d\bar{y}$  which translates to  $-4d^2\tau d^2y$ . Nevertheless we shall be able to check that our procedure agrees with the one used in [35, 36] including normalization. This is described at the end of §7.3.

field theory associated with the compact Calabi-Yau manifold, related to  $V_1$  via the action of superconformal generator  $T_F$ :

$$T_F(z)V_1(0) = -\frac{1}{z}\tilde{V}_1(0) + \text{non-singular} . \quad (7.16)$$

Using (7.15) we see that the contribution to (7.13) from the first term on the right hand side of (7.14) is given by

$$\begin{aligned} & \int d^2\tau \int d^2y \left\langle b_\tau \bar{b}_\tau \left[ \bar{c}(0)c(0) \left\{ \tilde{V}_1(0) - \frac{i}{2}k \cdot \psi(0)V_1(0) \right\} e^{ik \cdot X(0)} - \frac{1}{4}\bar{c}(0)\eta(0)e^{\phi(0)}V_1(0)e^{ik \cdot X(0)} \right] \right. \\ & \quad \left. \times \left( \mathcal{X}(\alpha y) + \alpha \partial\xi(\alpha y)c(y) \right) e^{-\phi(y)}V_2(y)e^{-ik \cdot X(y)} \right\rangle . \end{aligned} \quad (7.17)$$

As already mentioned above, and will be verified in §7.4, the total contribution is expected to be independent of  $\alpha$ . In order to make contact with the analysis of [35, 36] we shall now take the  $\alpha \rightarrow 1$  limit, which amounts to inserting the second picture changing operator at the location  $y$  of the second puncture. In this limit the  $c\partial\xi$  term from  $\mathcal{X}(\alpha y)$  cancels the  $\alpha \partial\xi(\alpha y)c(y)$  term. Thus the only term in the second line of (7.17) that contributes is the term

$$\lim_{\alpha \rightarrow 1} e^{\phi(\alpha y)}T_F(\alpha y)e^{-\phi(y)}V_2(y)e^{-ik \cdot X(y)} = \left\{ \tilde{V}_2(y) + \frac{i}{2}k \cdot \psi(y)V_2(y) \right\} e^{-ik \cdot X(y)} , \quad (7.18)$$

where  $\tilde{V}_2(y)$  is defined in the same way as  $\tilde{V}_1$  in (7.16)

$$T_F(z)V_2(0) = -\frac{1}{z}\tilde{V}_2(0) + \text{non-singular} . \quad (7.19)$$

Furthermore  $\phi$  charge conservation now allows us to drop the term proportional to  $e^{\phi(0)}$  from the first line of (7.17). Thus (7.17) now takes the form

$$\int d^2\tau \int d^2y \left\langle b_\tau \bar{b}_\tau \bar{c}(0)c(0) \left\{ \tilde{V}_1(0) - \frac{1}{2}ik \cdot \psi(0)V_1(0) \right\} e^{ik \cdot X(0)} \left\{ \tilde{V}_2(y) + \frac{1}{2}ik \cdot \psi(y)V_2(y) \right\} e^{-ik \cdot X(y)} \right\rangle . \quad (7.20)$$

This is precisely the term that was analyzed in [35, 36]. As emphasized there, if we work with strictly on-shell momentum  $k^2 = 0$  then the result vanishes. [35, 36] analyzed this by keeping the momenta slightly off-shell and at the end taking the  $k^2 \rightarrow 0$  limit. We shall come back to discuss this approach later, but for now we proceed by keeping  $k^2 = 0$  from the beginning. In that case this term does not contribute.

This leaves us with the contribution from the second term on the right hand side of (7.14), and we need to take the  $\alpha \rightarrow 1$  limit at the end. This can be analyzed by deforming the BRST

contour and picking up the contribution from the residues at the rest of the operators inserted in (7.13). Before doing that however we replace  $Q_B$  by its holomorphic part  $Q_B^R$  since only this part contributes to  $\{Q_B, \partial\xi\}$ . We have

$$[Q_B^R, \bar{c}(0)c(0)e^{-\phi(0)}V_1(0)e^{ik.X(0)}] = 0, \quad (7.21)$$

$$[Q_B^R, \mathcal{X}(\alpha y)] = 0, \quad (7.22)$$

$$[Q_B^R, \partial\xi(\alpha y)] = \partial\mathcal{X}(\alpha y), \quad (7.23)$$

$$[Q_B^R, c(y)e^{-\phi(y)}V_2(y)e^{-ik.X(y)}] = 0, \quad (7.24)$$

$$[Q_B^R, e^{-\phi(y)}V_2(y)e^{-ik.X(y)}] = \partial_y (c(y)e^{-\phi(y)}V_2(y)e^{-ik.X(y)}). \quad (7.25)$$

Finally  $\{Q_B^R, \bar{b}_\tau\}$  vanishes and  $\{Q_B^R, b_\tau\}$  generates total derivative with respect to  $\tau$  and integrates to zero. Thus the net contribution, after adding all the terms, is given by

$$\begin{aligned} & \int d^2\tau \int d^2y \left\langle b_\tau \bar{b}_\tau (\xi(u_1) - \xi(0)) \bar{c}(0)c(0)e^{-\phi(0)}V_1(0)e^{ik.X(0)} \right. \\ & \left. \left[ \mathcal{X}(\alpha y) \partial_y (c(y)e^{-\phi(y)}V_2(y)e^{-ik.X(y)}) + \alpha \partial\mathcal{X}(\alpha y) c(y)e^{-\phi(y)}V_2(y)e^{-ik.X(y)} \right] \right\rangle. \end{aligned} \quad (7.26)$$

This can be rewritten as

$$\int d^2\tau \int d^2y \partial_y \left[ \left\langle b_\tau \bar{b}_\tau \{ \xi(u_1) - \xi(0) \} \bar{c}(0)c(0)e^{-\phi(0)}V_1(0)e^{ik.X(0)} \mathcal{X}(\alpha y) c(y)e^{-\phi(y)}V_2(y)e^{-ik.X(y)} \right\rangle \right]. \quad (7.27)$$

Note that the only term of  $\mathcal{X}(\alpha y)$  that contributes is the one with  $\phi$  charge 2:

$$-\frac{1}{4}\partial\eta be^{2\phi} - \frac{1}{4}\partial(\eta be^{2\phi}). \quad (7.28)$$

Substituting this into (7.27) and using the operator product expansion to evaluate the  $\alpha \rightarrow 1$  limit, we get

$$-\frac{1}{4} \int d^2\tau \int d^2y \partial_y \left\langle b_\tau \bar{b}_\tau \{ \xi(u_1) - \xi(0) \} \bar{c}(0)c(0)e^{-\phi(0)}V_1(0)e^{ik.X(0)} \eta(y)e^{\phi(y)}V_2(y)e^{-ik.X(y)} \right\rangle. \quad (7.29)$$

This is a total derivative in  $y$ . Thus it can get a non-zero contribution only from the boundary near  $y = 0$  if the integrand has a singularity of the form  $1/\bar{y}$ . Now the only source of  $\bar{y}$  dependence in the above correlator is from the matter vertex operators  $V_1$  and  $V_2$ ; the  $e^{ik.X}$  factors can be ignored altogether since any contraction involving them will pick up  $k^2$  factors which vanish by on-shell condition. Thus we can simply replace  $V_1(0)V_2(y)$  by the part which

has a pole of the form  $1/\bar{y}$  and which has a non-vanishing expectation value on the torus. This is given by [35, 36]

$$V_1(0)V_2(y) = \frac{q}{\bar{y}}V_D(0), \quad (7.30)$$

where  $V_D$  is the dimension (1,1) operator representing the vertex operator of the auxiliary D-field associated with the anomalous U(1) gauge field and  $q$  is the charge carried by the vertex operator under this anomalous U(1).  $V_D$  is given by the product of the R-symmetry current on the right-moving sector of the world-sheet and the left-moving U(1) current associated with the anomalous U(1) gauge field. The other relevant operator products are

$$e^{-\phi(0)}e^{\phi(y)} \simeq -y + \mathcal{O}(y^2), \quad (7.31)$$

and

$$\{\xi(u_1) - \xi(0)\} \eta(y) \simeq y^{-1} + \mathcal{O}(y^0). \quad (7.32)$$

Substituting this into (7.29) we get

$$-\frac{q}{4} \int d^2\tau \langle \langle b_\tau \bar{b}_\tau \bar{c}(0) c(0) V_D(0) \rangle \rangle \times \int d^2y \partial_y \bar{y}^{-1}, \quad (7.33)$$

where it is understood that the integral over  $y$  is to be done by putting a cut-off  $|y| \geq \epsilon$  for some small positive number  $\epsilon$  – this corresponds to the infrared regularization  $s < \Lambda$  described in §3.8 with the identification  $|y| = e^{-s}$ ,  $\epsilon = e^{-\Lambda}$  – and we are supposed to pick up the boundary contribution from the  $|y| = \epsilon$  end. Writing the integral in terms of  $r = |y|$  and  $\theta = \text{Arg}(y)$  it is easy to see that the integral over  $y$  receives a contribution of  $-\pi$  from the boundary at  $|y| = \epsilon$ . Thus the final result for the two point function is given by

$$\frac{1}{4} \pi q \int d^2\tau \langle \langle b_\tau \bar{b}_\tau \bar{c}(0) c(0) V_D(0) \rangle \rangle. \quad (7.34)$$

This agrees with the result of [39] up to normalization. We shall check the normalization shortly.

Note that the entire extra contribution compared to that in [35, 36] for  $k^2 = 0$  came from the need to move the first picture changing operator from 0 to the position  $u_1$ , whose effect is given by the second term in the right hand side of (7.14). This is needed to ensure that the picture changing operators follow the correct arrangement in the degeneration limit.

In order to check that the analysis given above captures the complete result, we shall now verify that (7.34) gives the correct normalization and sign. We shall do this by comparing

the result with that of [35, 36] where the complete contribution came from (7.20) by keeping  $k$  slightly off-shell.<sup>15</sup> In this case the nonvanishing part of the result comes from keeping the second term inside each curly bracket in (7.20):

$$\frac{1}{4} \int d^2\tau \int d^2y \langle b_\tau \bar{b}_\tau \bar{c}(0) c(0) k \cdot \psi(0) V_1(0) e^{ik \cdot X(0)} k \cdot \psi(y) V_2(y) e^{-ik \cdot X(y)} \rangle . \quad (7.35)$$

Now by Lorentz invariance the  $\psi^\mu$ ,  $\psi^\nu$  correlator is proportional to  $\eta^{\mu\nu}$ , producing a factor of  $k^2$ . Since eventually we take the  $k^2 \rightarrow 0$  limit we must pick up the singular contribution proportional to  $1/k^2$  from the rest of the terms. This comes from the  $y \rightarrow 0$  limit of the integration. Eq.(7.30), (7.2) together with

$$e^{ik \cdot X(0)} e^{-ik \cdot X(y)} = |y|^{-k^2} + \text{less singular terms} \quad (7.36)$$

and the fact that  $V_1$ ,  $V_2$  anti-commute with  $\psi^\mu$  now give

$$-q \frac{k^2}{8} \int d^2\tau \int d^2y |y|^{-2-k^2} \langle b_\tau \bar{b}_\tau \bar{c}(0) c(0) V_D(0) \rangle . \quad (7.37)$$

In the  $k^2 \rightarrow 0$  limit the  $y$  integral produces a factor of  $-2\pi/k^2$ . Thus the net contribution is

$$\frac{1}{4} \pi q \int d^2\tau \langle b_\tau \bar{b}_\tau \bar{c}(0) c(0) V_D(0) \rangle . \quad (7.38)$$

This is in perfect agreement with (7.34).

## 7.4 $\alpha$ independence

Finally let us verify that the expression (7.13) is independent of  $\alpha$ . For this we take the  $\alpha$  derivative of this expression. Acting on  $\mathcal{X}(\alpha y)$  this generates a  $y\{Q_B^R, \partial\xi(\alpha y)\}$  while acting on  $\alpha\partial\xi(\alpha y)$  it gives  $\partial\xi(\alpha y) + \alpha y\partial^2\xi(\alpha y)$ . The BRST contour in  $\{Q_B^R, \partial\xi(\alpha y)\}$  can now be deformed.  $\{Q_B^R, \bar{b}_\tau\}$  vanishes and the residue from  $\{Q_B^R, b_\tau\}$  generates total derivatives in  $\tau$  which integrate to zero. On the other hand the  $\bar{c}(0)c(0)e^{-\phi(0)}V_1(0)e^{ik \cdot X(0)}$  is BRST invariant

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<sup>15</sup>Although we are reproducing the computation of [35, 36], we should keep in mind that this is different from the definition of off-shell amplitude we have given earlier. Refs. [35, 36] use a coordinate system in which the local coordinates at the punctures are taken to be  $u$  and  $u - y$  instead of scaling them by  $1/y$  as described in footnote 13. Nevertheless this computation was justified by showing that this produces correctly the location of the  $s$ -channel pole in an on-shell four point scattering amplitude. If instead we follow our approach then the off-shell computation will be very similar to the on-shell computation performed here. The  $y$  dependent scaling will remove the  $-k^2$  from the exponent of  $y$  in (7.37) and the final result will still come from boundary contributions.



and  $\{Q_B^R, e^{-\phi(y)} V_2(y) e^{-ik \cdot X(y)}\}$  generates  $\partial_y (c(y) e^{-\phi(y)} V_2(y) e^{-ik \cdot X(y)})$ . Combining this with the rest of the terms gives the  $\alpha$  derivative of (7.13) to be

$$\int d^2\tau \int d^2y \partial_y \left[ y \left\langle b_\tau \bar{b}_\tau \mathcal{X}(u_1) \bar{c}(0) c(0) e^{-\phi(0)} V_1(0) e^{ik \cdot X(0)} \partial \xi(\alpha y) c(y) e^{-\phi(y)} V_2(y) e^{-ik \cdot X(y)} \right\rangle \right]. \quad (7.39)$$

Since this is the integral of a total derivative, it is given by boundary contribution. Possible boundary terms could arise from around  $y = 0$  if the term inside the square bracket diverged as  $1/\bar{y}$  in this limit. Using (7.30), (7.31) we see however that it goes as  $y/\bar{y}$  in the  $y \rightarrow 0$  limit. Thus there are no non-zero boundary contributions, showing that (7.13) is indeed independent of  $\alpha$ .

## 7.5 Two loop dilaton tadpole

The Fayet-Iliopoulos term is also expected to generate a dilaton tadpole at two loop level. Formalism involving picture changing operators can also be used to compute this. This was done in [38]. Here we shall review the basic steps of [38] so that the reader can see the close parallel between the computation of one loop mass renormalization described above and the two loop dilaton tadpole.

For two loop one point function we need three insertions of picture changing operators. In the limit of degeneration to two tori, two of the picture changing operators must lie on the tori that contains the dilaton vertex operator in the  $-1$  picture while the third picture changing operators will have to lie on the torus without any external state. This was achieved in [38] by taking one of the picture changing operators on top of the dilaton vertex operator to bring it to a 0-picture vertex operator. This is convenient (and is analogous to taking the  $\alpha \rightarrow 1$  limit in the one loop analysis) but is not necessary. We shall proceed without taking this limit.

The second step involves expressing the  $-1$  picture dilaton vertex operator as  $\{Q_S, W\}$  where  $Q_S$  is the supersymmetry generator in the  $-1/2$  picture and  $W$  is the dilatino vertex operator in the  $-1/2$  picture. This can be expressed as the contour integral of the supersymmetry current around the dilatino vertex operator.

After summing over spin structures the correlation function involving the supersymmetry current satisfies the correct periodicity conditions on the genus two Riemann surface. In the third step we deform the contour of integration of the supersymmetry current away from the dilatino vertex operator and try to shrink it to a point. Naively one would expect that this should be possible leading to vanishing result, but it was found in [33, 38] that the correlation

function has spurious poles. Thus the result does not vanish, but can be expressed as the result of contour integration around the spurious poles. Let us denote by  $C$  the sum of all such contours.

In the next step we move the location of one of the picture changing operators leaving fixed the position of the contours. As a result the spurious poles shift. As long as we can ensure that the locations of all the spurious poles as a function of the location of the supersymmetry current move outside  $C$ , the final result will vanish. But in the process of moving the picture changing operator we pick up a contribution proportional to  $\mathcal{X}(z_1) - \mathcal{X}(\tilde{z}_1) = \{Q_B, \xi(z_1) - \xi(\tilde{z}_1)\}$  where  $z_1$  and  $\tilde{z}_1$  are the initial and final positions of the picture changing operator that is being moved. Note that in order that the term involving  $\mathcal{X}(\tilde{z}_1)$  vanish, we have to ensure that  $\tilde{z}_1$  is at a position in which the spurious poles are outside the contour  $C$  for all values of the moduli. In the degeneration limit this can be achieved if we ensure that  $\tilde{z}_1$  is on the ‘wrong side’, i.e. the side opposite to that of  $z_1$ . In contrast if  $z_1$  is on the same side as  $\tilde{z}_1$  in this limit then the spurious poles on the other side will be insensitive to  $\tilde{z}_1$  and continue to remain inside the contour  $C$ .

In the final step we deform the BRST contour in  $\{Q_B, \xi(z_1) - \xi(\tilde{z}_1)\}$  and express the result as a total derivative in the moduli space. The relevant boundary contribution comes from the degeneration limit described above. The contribution in the degeneration limit gives the expected contribution to the dilaton tadpole [38].

The reader would probably have noticed the close parallel between the one loop analysis of mass renormalization and the two loop analysis of the dilaton tadpole. In both cases we move a picture changing operator to the wrong side and show that the resulting contribution vanishes. Thus the result is given by the difference between inserting the picture changing operator on the right side and the wrong side. This in turn is a total derivative in the moduli space and receives contribution only from the boundary of the moduli space.

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